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SUPERSONIC FLUTTER OF A CIRCULAR CYLINDRICAL SHELL OF FINITE LENGTH IN AN AXISYMMETRICAL MODE†

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Abstract—In this paper the flutter in an axisymmetrical mode of a simply supported thin cylindrical shell is studied. The linearized Timoshenko shell equations and linear piston theory lead to a non-self-adjoint eigenvalue problem, which is investigated thoroughly and solved without further approximations. In this manner the critical speeds (flutter boundaries) are determined. Using these 'exact' results two checks can be made. A comparison of the exact results with those obtained by Galerkin's method proves the reliability of this method if applied to the present eigenvalue problem. On the other hand, a comparison of the exact result with experimental data will prove whether or not the physical assumptions upon which the eigenvalue is based are satisfactory.

NOTATION

A	$a_0^2\rho_0 M \dfrac{12(1-\nu^2)}{E}\left(\dfrac{R}{h}\right)^3\left(\dfrac{L}{R}\right)^3$, generalized velocity; equation (18b), $A \geq 0$
A_{crit}	critical value of A, see (24)
$\dot{A}_{2m}$	for $A = \dot{A}_{2m}$ the eigenvalues $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ coincide
A_{2m}^*	$80m^3\pi^3/(3\sqrt{6})$; equation (80)
a_0	velocity of sound of the undisturbed airstream
B	$12(1-\nu^2)\left(\dfrac{L}{R}\right)^4\left(\dfrac{R}{h}\right)^4\left(1+\dfrac{\nu p_i}{E}\dfrac{R}{h}\right)$; equation (18c)
$D(A, \lambda)$	matrix, see equation (43b)
E	Young's modulus
$F(\alpha, \beta)$	see equation (66)
$F^*(A, \alpha)$	see equation (68)
$g(z_1, z_2, z_3, z_4)$	denominator of Δ_3, see equation (59c)
h	thickness of the shell
i	$\sqrt{-1}$
L	length of the shell
M	Mach number, $\dfrac{U}{a_0}$
M_{crit}	critical Mach number
$\overline{N_x}, \overline{N_y}$	stress resultants due to internal pressure differential p_i, force per unit length, see equations (1) and (12)
$N(A, \mathfrak{C})$	number of zeros of $\Delta_1(A, \lambda)$ encircled by $\mathfrak{C}$; equation (52)
p	external aerodynamic pressure due to the disturbance of the airstream, positive in the positive z-direction
$p(x, t)$	$-a_0\rho_0\left(U\dfrac{\partial\hat{w}}{\partial x}+\dfrac{\partial\hat{w}}{\partial t}\right)$, approximation for p according to linear piston theory; equation (7)
$p_0(x, t)$	$-a_0\rho_0 U\dfrac{\partial\hat{w}}{\partial t}$, approximation for p neglecting the aerodynamic damping; equation (94)

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p_i	internal static pressure differential, positive if the internal pressure is higher than the outside one
$p(z, A, \lambda) = 0$	auxiliary equation; equation (57)
R	radius of the shell
t	time
U	velocity of the undisturbed external airstream
U_{crit}	flutter speed
$\hat{u}, \hat{v}, \hat{w}$	components of displacement of a point of the shell from the uniform static stress status
$w^*(x)$	mode shape of $\hat{w}(x, t)$, see equation (13)
$w(\xi)$	mode shape, see equation (17)
$\tilde{w}(\xi)$	comparison function used for Galerkin's method, see equation (91)
$w_n(\xi, A)$	eigenfunction of the eigenvalue problem (19) belonging to the eigenvalue $\lambda_n(A)$
$w(\xi, A, \lambda)$	solution of the differential equation (19a)
x, y, z	curvilinear coordinate system, see Fig. 1
z	complex number, equation (57)
z_1, z_2, z_3, z_4	roots of the auxiliary equation (57)
$z_{n1}, \ldots, z_{n4}$	roots of the auxiliary equation (70)
α	$\frac{1}{2}(z_1 + z_2)$; equation (63a)
$\alpha_n(A)$	$\frac{1}{2}[z_{n1}(A) + z_{n2}(A)]$; equation (73)
β	$\frac{1}{2i}(z_1 - z_2)$; equation (63a)
$\beta_n(A)$	$\frac{1}{2i}[z_{n1}(A) - z_{n2}(A)]$; equation (73)
$(\alpha_{2m-1}(A), \beta_{2m-1}(A))$ $(\alpha_{2m}(A), \beta_{2m}(A))$	intersections of the curve $\mathfrak{C}_A^{\mp}$ with the loop l_{2m}, $m = 1, 2, \ldots$; $0 < A \leq \mathring{A}_{2m}$, see equations (83)–(85)
β	damping, $\beta \geq 0$, see equations (8), (9), (11)
β_{crit}	critical damping, see equation (10)
γ	damping coefficient, see equation (9); $\gamma \geq 0$
$\Delta_1(A, \lambda)$	$\det[D(A, \lambda)]$, equation (43a)
$\Delta_3(z_1, z_2, z_3, z_4)$	determinant, see equation (59b)
$\delta, \delta', \delta_n, \varepsilon, \varepsilon'$	positive constants
λ^*	$12(1 - \nu^2)\frac{L^4}{Eh^3}[\rho_s h\omega^2 - i(a_0\rho_0 + \beta)\omega]$, eigenvalue parameter, see equation (18d)
λ	$\lambda^* - B$, eigenvalue parameter, see equation (18e)
$\lambda_n^*(A)$	$\lambda_n(A) + B$
$\lambda_n(A)$	nth eigenvalue of the eigenvalue problem (19), enumeration according to equation (56), $n = 1, 2, \ldots$
μ	$\lambda = \mu$ root of the equation $\Delta_1(A, \lambda) = 0$
$\mu_n(A)$, $n = 1, 2, \ldots$	nth root of the equation $\Delta_1(A, \lambda) = 0$, enumeration according to Section 2.5
$\hat{\mu}(A)$	fictive root of the equation $\Delta_1(A, \lambda) = 0$, used temporarily in Section 2.5
ν	Poisson's ratio, also index
ξ	$\frac{x}{L}$, see equation (16)
ρ_0	density of the undisturbed air
ρ_s	density of the shell material
φ	angle, $y = R\varphi$
$\varphi_\nu(\xi)$	$\sqrt{2}\sin(\nu\pi\xi)$, $\nu = 1, 2, \ldots$; equation (34)
$\psi_{1m}(\xi, A, \lambda)$ $m = 1, 2, 3, 4$	fundamental system, see equations (40) and (41)
$\psi_{2m}(\xi, n^4\pi^4)$ $m = 1, 2, 3, 4$	fundamental system, see equations (51)
$\psi_{3m}(\xi, A, \lambda)$, $m = 1, 2, 3, 4$	fundamental system, see equations (58)
ω	circular frequency, radians per second; equation (13)
$\omega_{n1}(A), \omega_{n2}(A)$	see equation (20)
$\mathfrak{C}$	curve in the complex λ-plane; equation (52)
$\mathfrak{C}[\varepsilon', \mu_{n_0}(A)]$	circle defined by (53)
$\mathfrak{C}_A$	curve in the real $\alpha\beta$-plane, defined by $A = 4\alpha(\alpha^2 - \beta^2)$
$\mathfrak{C}_A^{+}, \mathfrak{C}_A^{\mp}, \mathfrak{C}_A^{=}$	branches of the curve $\mathfrak{C}_A$ for $A > 0$
l_{2m}, $m = \pm 1, \pm 2, \ldots$	loops in the real $\alpha\beta$-plane defined by $F(\alpha, \beta) = 0$

INTRODUCTION

DURING the last years the problem of the flutter of thin cylindrical shells has attracted much attention. Until recently little experimental work was done in this field. A detailed survey of the theoretical methods and results is given by Fung in [1]. All theoretical investigations use approximations. On the physical side different approximate shell equations and unsteady aerodynamic pressure were used, on the mathematical side in general Galerkin's method was used. Galerkin's method is employed as a handy tool in almost all vibration problems occurring in engineering. But it is a fact that the reliability of Galerkin's method has been proved only for a certain class of problems. There are cases known in which this method led to spurious results. Therefore it is a question if Galerkin's method can be used for handling the present flutter problem. In particular, if the theoretical results just mentioned disagree with experimental results, one must determine which of the approximations are to be blamed.

In a program pursued at the Guggenheim Aeronautical Laboratory of the California Institute of Technology, experimental flutter tests and theoretical analyses were linked together, see [2] and [3]. A rigorous mathematical analysis was made for two purposes: (1) to assess physical assumptions by comparing mathematical and experimental results, (2) to assess the accuracy of the Galerkin's method when applied to this problem. The present paper is devoted to these theoretical investigations.

In the following, the flutter of a simply supported circular cylindrical shell of finite length in an axisymmetrical mode is investigated. The shell is exposed externally to a supersonic airstream parallel to the generators of the cylinder and internally to a constant pressure. The restriction to axisymmetrical flutter modes was done for the sake of simplicity. However, there are hints from previous investigations that the symmetric flutter mode is possibly one of the most dangerous modes, see e.g. [1] and the introduction of [4].

The flutter problem deals with the following question. Does there exist for the shell under investigation a special velocity of the outer airstream, which is called the 'critical speed' or the 'flutter speed', with the following properties:

(1) If the velocity of the airstream equals the critical speed, the shell is capable of harmonic oscillations.
(2) For all stream velocities below the critical speed all possible oscillations with an exponential time factor are positively damped, i.e. the amplitude decreases (exponentially) as time goes on.
(3) For velocities beyond the critical speed there exist divergent oscillations with exponentially increasing amplitude as time goes on. The mode shape of those undamped or negatively damped vibrations, mentioned in (1) and (3) respectively, is called a 'flutter mode'.

It is obvious that the existence of a flutter condition is of concern to the practical design of an aircraft.

In the present analysis, the following physical assumptions are employed. The elastic forces are represented by the linearized Timoshenko equations. The inertia forces of the shell elements in the direction of the shell generators are neglected. The external air forces are represented by the linear piston theory expression. The acoustic pressure of the still air on the inside of the cylinder is neglected. These assumptions finally lead to an eigenvalue problem of an ordinary differential equation of the fourth order, which is in general non-self-adjoint (see Section 1). The governing parameter, called the 'generalized velocity', is proportional to the velocity of the external airstream. The question whether or not there exist for a certain stream velocity undamped or negatively damped oscillations of the shell (mentioned under (1) and (3), respectively) is equivalent to the question whether or not the corresponding eigenvalue problem possesses eigenvalues located on the boundary or on the outside of a certain parabola, called the 'stability parabola'. If for a certain generalized velocity all the eigenvalues are located on the inside of this parabola, only positively damped oscillations are possible. So we have to investigate the dependence of the set of eigenvalues upon the generalized velocity.

If the air velocity is zero, the eigenvalue problem under consideration is self-adjoint, its eigenvalues are well-known, all of them are located inside the stability parabola. We will trace the variation of these eigenvalues for increasing values of the generalized velocity. This is done in Sections 2 and 3. It should be noted that Movchan also investigated in [5–7] this eigenvalue problem to some extent. Partially we will use his methods, but on certain aspects we will use another approach which exhibits greater details. Furthermore, Movchan gives results only for a range of generalized velocity which is far too small for our purpose. The relations between Movchan's analysis and the present one will be discussed in Section 5.

In Section 3 the eigenvalues will be evaluated for values of the generalized velocity up to 110,000. Those eigenvalues will be compared with those obtained Galerkin's method. In principle, Galerkin's method proves to be satisfactory.

In Section 4 we evaluate the flutter speed for a cylindrical shell, using the results of Section 3. In general, the chosen parameter values, e.g. thickness, radius and length of the cylinder etc. are close to the values of a test cylinder reported in [2]. The influence of the material and aerodynamic damping will be investigated. The results demonstrate that both of them have a considerable stabilizing effect. Further, the rate of divergence at flutter conditions will be treated. In the frame of the present analysis the influence of the internal pressurization is very small. The flutter speeds obtained by the present analysis are larger than those obtained in some previous investigations, as a comparison with results of Holt and Strack shows.

In the meantime experiments were performed at the Ames Research Center Wind Tunnel with the above mentioned copper shell, see [2, 3]. No flutter was observed in the Mach number range predicted by the present analysis. This indicates that the basic physical assumptions have to be improved. The approximation of the aerodynamic pressure by the linear piston theory expression, the omission of boundary layer effects, and the neglect of the inertia forces of the shell elements in the longitudinal direction might be responsible, at least partially, for this disagreement.

1. FORMULATION OF THE PROBLEM

We consider in the following the axisymmetric vibrations with an exponential time factor of a simply supported (circular) cylindrical shell of finite length, which is exposed externally to a supersonic uniform airstream, and internally to a constant pressure. Let a curvilinear coordinate system be chosen, as shown in Fig. 1. The x-axis is directed along the generators of the cylinder, the y-axis, $y = R\varphi$, is measured clockwise in the circumferential direction, and the z-axis is directed inward along the normal to the middle surface of the shell. The length of the shell is denoted by L, the radius by R, the wall thickness by

h, Young's modulus, Poisson's ratio and the density of the shell material are denoted by E, ν, and ρ_s respectively. The shell is assumed to be very thin, i.e. $h \ll R$ holds. The air-stream is parallel to the generators of the cylinder, its velocity is U, positive in the positive x-direction. The density of the air and the sound velocity are denoted by ρ_0 and a_0, respectively.

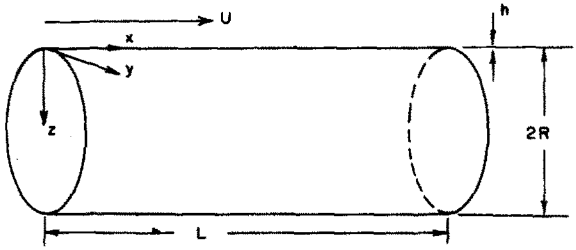

FIG. 1. Coordinate system.

Let $\overline{N_x}$ and $\overline{N_y}$ be the stress resultants in the x and y directions, introduced by the internal static pressure differential p_i. For simplicity, we shall assume that

$$\overline{N_x} = \text{const}, \qquad \overline{N_y} = \text{const}, \tag{1}$$

whereas all other steady state stress resultants and couples vanish. The effect of any possible bending caused by the end conditions will be neglected. Let $\hat{u}$, $\hat{v}$, $\hat{w}$ be the vibratory displacements, i.e. the deviations from the uniform state of stress of the unpressurized cylinder or eventually from the uniform state of stress induced by the internal static pressure differential p_i. $\hat{u}$, $\hat{v}$, $\hat{w}$ and their spatial derivatives are assumed to be of infinitesimal amplitude.

In the following, the basic assumptions of Love's first approximation (see Timoshenko, [8], Chap. II) will be used. The resulting equations are nonlinear. According to the above-mentioned assumptions small quantities of second and higher order will be neglected, as already has been done in [9]. Because we are interested in axisymmetrical vibration modes only, we put $\hat{v}$ and all derivatives in respect to y (or φ) equal to zero. In this way we obtain from the three equations (4) of [9] only two equations for $\hat{u}(x, t)$ and $\hat{w}(x, t)$:

$$\frac{\partial^2 \hat{u}}{\partial x^2} - \frac{\nu}{R}\frac{\partial \hat{w}}{\partial x} + \overline{N_y}\frac{1-\nu^2}{EhR}\frac{\partial \hat{w}}{\partial x} - \frac{1-\nu^2}{Eh}\rho_s h \frac{\partial^2 \hat{u}}{\partial t^2} = 0, \tag{2}$$

$$\nu\frac{\partial \hat{u}}{\partial x} - \frac{\hat{w}}{R} - \frac{Rh^2}{12}\frac{\partial^4 \hat{w}}{\partial x^4} + \frac{R(1-\nu^2)}{Eh}\left(\overline{N_x}\frac{\partial^2 \hat{w}}{\partial x^2} + p - \rho_s h \frac{\partial^2 \hat{w}}{\partial t^2}\right) = 0. \tag{3}$$

p denotes the external aerodynamic pressure due to the disturbance of the airstream by the shell vibration; p is measured positive in the positive z-direction. The assumption that the cylinder is simply supported on the ends leads to the boundary conditions

$$\left(\frac{\partial \hat{u}}{\partial x}\right)_{x=0} = \left(\frac{\partial \hat{u}}{\partial x}\right)_{x=L} = 0; \tag{4a}$$

$$\hat{w}(0, t) = \hat{w}(L, t) = 0; \qquad \left(\frac{\partial^2 \hat{w}}{\partial x^2}\right)_{x=0} = \left(\frac{\partial^2 \hat{w}}{\partial x^2}\right)_{x=L} = 0. \tag{4b, c}$$

In the following, we make the additional assumption that the inertia forces of the shell particles in the x-direction are negligible. Then we obtain, using equation (2) and the boundary conditions (4a) and (4b)

$$\frac{\partial \hat{u}}{\partial x} = \left(\frac{\nu}{R} - \frac{1-\nu^2}{EhR}\overline{N_y}\right)\hat{w}(x,t). \tag{5}$$

Substituting (5) into equation (3) one arrives at

$$-\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 \hat{w}}{\partial x^4} + \overline{N_x}\frac{\partial^2 \hat{w}}{\partial x^2} - \frac{1}{R^2}\left(Eh + \nu\overline{N_y}\right)\hat{w} + p = \rho_s h \frac{\partial^2 \hat{w}}{\partial t^2}, \tag{6}$$

which is an equation for $\hat{w}(x,t)$ only. Again the boundary conditions are given by (4b) and (4c). Because of equation (5) the function $\hat{u}(x,t)$ is known as soon as $\hat{w}(x,t)$ is known. Therefore, in the following, we have only to deal with $\hat{w}(x,t)$.

For the external aerodynamic pressure p the approximation given by the linear piston theory (see e.g. [10] and [11]) will be used, i.e.

$$p(x,t) = -a_0\rho_0\left(U\frac{\partial \hat{w}}{\partial x} + \frac{\partial \hat{w}}{\partial t}\right). \tag{7}$$

The material damping is assumed to be of the viscous type. In order to include this damping force, we add to the right-hand side of equation (6) a term $\beta(\partial\hat{w}/\partial t)$ ($\beta \geq 0$) and obtain

$$-\frac{Eh^3}{12(1-\nu^2)}\frac{\partial^4 \hat{w}}{\partial x^4} + \overline{N_x}\frac{\partial^2 \hat{w}}{\partial x^2} - \frac{1}{R^2}\left(Eh + \nu\overline{N_y}\right)\hat{w} - a_0\rho_0\left(U\frac{\partial \hat{w}}{\partial x} + \frac{\partial \hat{w}}{\partial t}\right) = \rho_s h \frac{\partial^2 \hat{w}}{\partial t^2} + \beta\frac{\partial \hat{w}}{\partial t}. \tag{8}$$

Usually the damping β is given in fractions of the critical damping β_{crit},

$$\beta = \gamma\beta_{\text{crit}} \tag{9}$$

where γ is the 'damping coefficient'. For the thin shells under consideration one can use (for details see e.g. [12])

$$\beta_{\text{crit}} = 2\frac{h}{R}\sqrt{(E\rho)_s},^{\dagger} \tag{10}$$

which leads to

$$\beta = 2\gamma\frac{h}{R}\sqrt{(E\rho)_s}. \tag{11}$$

So far as the stress resultants $\overline{N_x}$ and $\overline{N_y}$ are concerned, we are only interested in such stresses caused by the pressurizing of the cylinder with an internal static pressure differential p_i, positive if the internal pressure is higher than the outside (static) pressure. According to the properties of the experimental model mentioned in the Introduction, one can use as a fair approximation $\overline{N_x} = 0$. So we have

$$\overline{N_x} = 0, \qquad \overline{N_y} = p_i R. \tag{12}$$

† The critical damping for a spring-dashpot system is treated e.g. in [13], Chap. I and in [14], Appendix 3. In [12] it has been shown that for thin shells vibrating in vacuum in the form $w^*(x,t) = w_0 \sin(\pi m x/L)\exp(i\omega^*_{m0}t)$ the critical damping $\beta_{\text{crit}_{m0}}$ is given by $\beta_{\text{crit}_{m0}} = 2\rho_s h\omega_{m0}$, where ω_{m0} is the natural frequency of the undamped shell freely vibrating in vacuum with the same mode pattern. For small values of m it is in fair approximation $(\omega_{m0})^2 = E/(R^2\rho_s)$.

Now we consider solutions $\hat{w}(x, t)$ of equation (8) of the form

$$\hat{w}(x, t) = w^*(x)e^{i\omega t}, \tag{13}$$

which satisfy the boundary conditions (4b) and (4c). As soon as there exists such a solution (13) with

$$\mathrm{Im}(\omega) < 0 \tag{14}$$

the vibration $\hat{w}(x, t)$ is unstable, i.e. the amplitude is increasing with increasing time, i.e. the cylinder flutters. For

$$\mathrm{Im}(\omega) \geq 0 \tag{15}$$

the corresponding solution (13) is stable or neutrally stable. It is the purpose of the following investigations to find special values U_{crit} of the airstream velocity U with the following properties: Having fixed all parameters besides U, one considers the solutions of equation (8) which satisfy the boundary conditions. For all U located in the interval $0 \leq U \leq U_{\text{crit}}$ the corresponding frequencies ω satisfy the inequality (15), whereas in any right side neighborhood of U_{crit} there exist values U, for which at least one solution (13) possesses an ω which satisfies the inequality (14). This critical value U_{crit} is called the 'critical velocity' or the 'flutter speed'.

Substituting (13) into (8), introducing the non-dimensional variable

$$\xi = \frac{x}{L}, \tag{16}$$

putting

$$w^*(x) = w(\xi), \tag{17}$$

using equations (12) and introducing

$$M = \frac{U}{a_0}, \qquad (U \geq 0) \tag{18a}$$

$$A = a_0^2 \rho_0 M \frac{12(1-\nu^2)}{E}\left(\frac{R}{h}\right)^3\left(\frac{L}{R}\right)^3, \text{ (generalized velocity)} \tag{18b}$$

$$B = 12(1-\nu^2)\left(\frac{L}{R}\right)^4\left(\frac{R}{h}\right)^2\left(1+\frac{\nu p_i}{E}\frac{R}{h}\right), \tag{18c}$$

$$\lambda^* = 12(1-\nu^2)\frac{L^4}{Eh^3}[\rho_s h\omega^2 - i(a_0\rho_0+\beta)\omega], \tag{18d}$$

$$\lambda = \lambda^* - B, \tag{18e}$$

one arrives at

$$\frac{d^4 w}{d\xi^4} + A\frac{dw}{d\xi} = (\lambda^* - B)w(\xi) = \lambda w(\xi), \qquad 0 \leq \xi \leq 1, \tag{19a}$$

$$w(0) = w(1) = \left(\frac{d^2 w}{d\xi^2}\right)_{\xi=0} = \left(\frac{d^2 w}{d\xi^2}\right)_{\xi=1} = 0. \tag{19b}$$

This is an eigenvalue problem with the eigenvalue parameter λ^* or λ. As soon as A is unequal to zero—we are restricted to real, non-negative values of A—this problem is a non-self-adjoint one.

If we put in (19) $B = 0$, i.e. $R = \infty$, then we obtain the eigenvalue problem which governs the flutter of a simply supported flat plate.

For a time we shall consider the whole problem exclusively from the mathematical point of view, irrespective of the fact that the eigenvalue problem (19) may have no physical meaning for small values of the parameter A. (For values of M below $\sqrt{2}$ the approximation of the aerodynamic pressure by the linear piston theory expression is unreasonable.) Especially we shall investigate the dependence of the eigenvalues $\lambda^* = \lambda_n^*(A)$ or $\lambda_n = \lambda_n(A)$ of the eigenvalue problem (19) upon the (real) parameter A, having fixed all other parameters.† Later, when the results of these investigations are to be applied for establishing the flutter boundaries, we shall restrict A to reasonably large values.

For any eigenvalue $\lambda_n^*(A)$ there are given by equation (18d) two frequencies $\omega_{n1}(A)$ and $\omega_{n2}(A)$

$$\omega_{n1,2}(A) = i\frac{a_0\rho_0+\beta}{2\rho_s h}\left\{1 \pm \sqrt{\left[1-\lambda_n^*(A)\frac{E\rho_s}{3(1-\nu^2)(a_0\rho_0+\beta)^2}\left(\frac{h}{R}\right)^4\left(\frac{R}{L}\right)^4\right]}\right\}. \tag{20}$$

In any case, notwithstanding the value of $\lambda_n^*(A)$, there is at least one of these two frequencies, say $\omega_{n1}(A)$, with

$$\mathrm{Im}[\omega_{n1}(A)] > 0. \tag{21}$$

The second frequency, $\omega_{n2}(A)$, satisfies the condition

$$\mathrm{Im}[\omega_{n2}(A)] > 0 \quad \text{or} \quad \mathrm{Im}[\omega_{n2}(A)] = 0 \quad \text{or} \quad \mathrm{Im}[\omega_{n2}(A)] < 0 \tag{22}$$

if and only if $\lambda_n^*(A)$ is located on the inside, on the boundary or on the outside of the so-called 'stability parabola' as defined by

$$Re(\lambda^*) = \frac{E\rho_s}{12(1-\nu^2)(a_0\rho_0+\beta)^2}\left(\frac{R}{L}\right)^4\left(\frac{h}{R}\right)^4[\mathrm{Im}(\lambda^*)]^2 \tag{23a}$$

respectively, as one easily verifies‡. Using the relation (18e) the equation for the stability parabola can be written also in the form

$$Re(\lambda)+B = \frac{E\rho_s}{12(1-\nu^2)(a_0\rho_0+\beta)^2}\left(\frac{R}{L}\right)^4\left(\frac{h}{R}\right)^4[\mathrm{Im}(\lambda)]^2. \tag{23b}$$

So, having fixed all other parameters, we are mainly concerned in finding the critical value of the parameter A, A_{crit}, with the following properties:

(1) For all values of A with $0 \leq A < A_{\mathrm{crit}}$ all eigenvalues $\lambda_n(A)$ of the eigenvalue problem (19) are located on the inside of the stability parabola (23b).

(2) For $A = A_{\mathrm{crit}}$ at least one of the eigenvalues $\lambda_n(A_{\mathrm{crit}})$ is located on the boundary of the stability parabola, whereas none of the eigenvalues $\lambda_n(A_{\mathrm{crit}})$ is located on the outside of the stability parabola.

(24)

† Using this notation we anticipate that for any A under consideration the corresponding set of eigenvalues is countable, which will be proved later in Section 2.

‡ A corresponding stability parabola has already been introduced by Movchan in [5].

(3) In any right-side neighborhood of A_{crit} there exists at least one value of A, for which at least one of the eigenvalues $\lambda_n(A)$ is located on the outside of the stability parabola. (24) (Cont'd)

(Hence, for the determination of A_{crit} some knowledge is needed of *all* eigenvalues $\lambda_n(A)$. Therefore it is obvious that Galerkin's method alone, even if it would furnish reasonable approximations for the corresponding eigenvalues, cannot be suitable for the evaluation of A_{crit}.) For this purpose we shall investigate in Sections 2 and 3 some properties of the eigenvalues $\lambda_n(A)$, especially with respect to their dependence upon the parameter A. It will be shown that the eigenvalues depend continuously upon A, so that one can trace them, starting with $A = 0$, for which the eigenvalues are well-known. In Section 2.1 it will be proved that all eigenvalues have a positive real part. Therefore only non-real eigenvalues can possibly be located on the outside of the stability parabola (23b). For this reason we are most interested in criteria which enable us to decide for a certain value of A, which eigenvalues are possibly non-real and which eigenvalues are certainly real.

As already has been mentioned in the Introduction, these considerations will to some extent repeat those outlined by Movchan (see [5–7]), but they will partially deviate from Movchan's line and go into more detail. The relations between Movchan's analysis and the present one will be discussed in Section 5.

2. INVESTIGATION OF THE EIGENVALUE PROBLEM

In this section we will investigate some features of the eigenvalue problem (19) in the form $w^{(4)}(\xi)+Aw^{(1)}(\xi) = \lambda w(\xi)$. For $A = 0$, as is well-known, the eigenvalues $\lambda_n(0)$ are given by

$$\lambda_n(0) = n^4\pi^4, \qquad n = 1, 2, \ldots \tag{25}$$

For $A > 0$ the eigenvalue problem (19) is no longer a self-adjoint one, therefore it is possible that for $A > 0$ there exist non-real eigenvalues. If λ_0 is an eigenvalue of (19), then $\bar{\lambda}_0$ is also an eigenvalue because the coefficients of (19a) and the boundary conditions (19b) are real†.

2.1. *All eigenvalues possess a positive real part*

Let $\lambda_n(A)$ be an eigenvalue of the eigenvalue problem (19), let $w_n(\xi, A)$ be a corresponding eigenfunction. Without restriction we can assume

$$\int_0^1 |w_n(\xi, A)|^2 \, d\xi = 1. \tag{26}$$

Multiplying the differential equation (19a) for $w_n(\xi, A)$ by the function $\overline{w_n(\xi, A)}$, using integration by parts and the boundary conditions, one obtains

$$\lambda_n(A) = \int_0^1 |w_n^{(2)}|^2 \, d\xi + A\int_0^1 w_n^{(1)}\overline{w_n} \, d\xi. \tag{27}$$

† Here and in the following the barred symbol denotes the conjugate complex number, i.e. $\overline{a+ib} = a-ib$; (a, b real).

Using again integration by parts and the boundary conditions we find immediately that

$$\int_0^1 w_n^{(1)}\overline{w_n}\,d\xi$$

is a pure imaginary number. Since A is a real number, we obtain

$$Re[\lambda_n(A)] = \int_0^1 |w_n^{(2)}|^2\,d\xi. \tag{28}$$

Because of the boundary conditions, $w_n(\xi, A)$ cannot be a linear function of ξ, the right-hand side of (28) is positive and therefore

$$Re[\lambda_n(A)] > 0. \tag{29}$$

This statement can also be drawn from Movchan's consideration, mentioned in Section 2 of [6].

2.2. *An inequality for the eigenvalues $\lambda_n(A)$*†

In this section we will prove that for any given A ($A \geq 0$) all eigenvalues $\lambda_n(A)$, $n = 1, 2, \ldots$ of the eigenvalue problem (19) are located in the union set of the circles

$$|\lambda - \pi^4 n^{*4}| \leq A(n^*+1)\pi, \qquad n^* = 1, 2, \ldots \tag{30}$$

For this purpose let $w_n(\xi, A)$ be an eigenfunction, which satisfies the relation (26) and belongs to the eigenvalue $\lambda_n(A)$. Using integration by parts and the boundary conditions, we obtain

$$\int_0^1 w_n^{(1)}\overline{w_n^{(1)}}\,d\xi = -\int_0^1 w_n^{(2)}\overline{w_n}\,d\xi. \tag{31}$$

Using (31), (26) and Schwarz's inequality one arrives at

$$\left(\int_0^1 |w_n^{(1)}|^2\,d\xi\right)^2 = \left(\int_0^1 w_n^{(2)}\overline{w_n}\,d\xi\right)^2 \leq \int_0^1 |w_n^{(2)}|^2\,d\xi. \tag{32}$$

Employing the differential equation (19a) and the relations (28) and (32) we obtain

$$\int_0^1 |w_n^{(4)} - \lambda_n(A)w_n|^2\,d\xi = A^2\int_0^1 |w_n^{(1)}|^2\,d\xi \leq A^2\left(\int_0^1 |w_n^{(2)}|^2\,d\xi\right)^{\frac{1}{2}} = A^2\left(Re[\lambda_n(A)]\right)^{\frac{1}{2}}. \tag{33}$$

Let $\varphi_\nu(\xi)$, $\nu = 1, 2, \ldots$ be the normalized and mutually orthogonal eigenfunctions of the eigenvalue problem (19) for $A = 0$ (for $A = 0$ the eigenvalue problem is self-adjoint),

$$\phi_\nu(\xi) = \sqrt{(2)}\sin(\nu\pi\xi), \qquad \nu = 1, 2, \ldots \tag{34}$$

The boundary conditions (19b) are independent of A. Therefore the eigenfunction $w_n(\xi, A)$ can be expended in an absolutely and uniformly convergent series (see e.g. [15], p. 360)

$$w_n(\xi, A) = \sum_{\nu=1}^{\infty} c_{n\nu}\varphi_\nu(\xi), \qquad 0 \leq \xi \leq 1. \tag{35}$$

† The author is indebted to Professor Jürgen Moser, New York University, for the disclosure of the inequality (30) and its proof.

This series can be differentiated piecewise four times†. In this way we learn from (33)

$$A^2\{Re[\lambda_n(A)]\}^{\frac{1}{2}} \geq \int_0^1 |w_n^{(4)} - \lambda_n(A)w_n|^2 \, d\xi$$
$$= \sum_{\nu=1}^{\infty} |c_{n\nu}|^2 \cdot |(\nu\pi)^4 - \lambda_n(A)|^2 \geq \min_{\nu} |(\nu\pi)^4 - \lambda_n(A)|^2. \tag{36}$$

Let n^* be that value of ν for which the minimum is achieved,

$$\min_{\nu} |(\nu\pi)^4 - \lambda_n(A)|^2 = |(n^*\pi)^4 - \lambda_n(A)|^2. \tag{37}$$

Then it holds obviously

$$Re[\lambda_n(A)] \leq [(n^*+1)\pi]^4. \tag{38}$$

Combining (36), (37), and (38), one finally arrives at

$$|\lambda_n(A) - (n^*\pi)^4| \leq A\{Re[\lambda_n(A)]\}^{\frac{1}{4}} \leq A(n^*+1)\pi. \tag{39}$$

The relation (39), where the positive integer n^* depends upon n and A, holds for any index n and any value of A ($A \geq 0$), and therefore our statement has been proved.

This property of the eigenvalues $\lambda_n(A)$ will be used later in Section 2.5. Especially we will take advantage of the fact that for a given value A and for sufficiently large n^* the circles (30) are separated.

2.3. *Characteristic equation*

We choose a fundamental system

$$\psi_{1m}(\xi) = \psi_{1m}(\xi, A, \lambda), \qquad m = 1, 2, 3, 4 \tag{40}$$

of the differential equation (19a), which satisfies, independently of A and λ, the initial conditions

$$\psi_{1m}^{(k)}(0) = \delta_{m,k+1}, \qquad k = 0, 1, 2, 3; \qquad m = 1, 2, 3, 4, \tag{41}$$

where $\delta_{m,k+1}$ is the Kronecker symbol. These four linearly independent solutions $\psi_{1m}(\xi, A, \lambda)$ are regular analytic functions in ξ, A and λ, see e.g. [17]. Any solution $w(\xi, A, \lambda)$ of (19a) can be represented as a linear combination of these four functions

$$w(\xi, A, \lambda) = \sum_{m=1}^{4} c_m \psi_{1m}(\xi, A, \lambda). \tag{42}$$

A special number (real or complex) $\lambda = \lambda_0$ is an eigenvalue of the eigenvalue problem (19) if and only if there exists a corresponding solution $w(\xi, A, \lambda_0) \not\equiv 0$ of (19a) which satisfies the boundary conditions (19b). This happens if and only if $\lambda = \lambda_0$ is a zero of

† The function

$$W(\xi) = \sum_{\nu=1}^{\infty} \sqrt{(2)}c_{n\nu} \sin(\nu\pi\xi)$$

is periodic with the period 2, it is $W(\xi) = w_n(\xi, A)$ for $0 \leq \xi \leq 1$. The function $w_n(\xi, A)$, being a solution of the differential equation (19a) can be differentiated to any order, hence the same holds for $W(\xi)$. Therefore the Fourier expansion of $W(\xi)$ can be differentiated to any order term by term, see e.g. [16].

the determinant

$$\Delta_1(A, \lambda) = \det[D(A, \lambda)], \tag{43a}$$

where $D(A, \lambda)$ is the matrix

$$D(A, \lambda) = \begin{pmatrix} \psi_{11}(0) & \psi_{12}(0) & \psi_{13}(0) & \psi_{14}(0) \\ \psi_{11}^{(2)}(0) & \psi_{12}^{(2)}(0) & \psi_{13}^{(2)}(0) & \psi_{14}^{(2)}(0) \\ \psi_{11}(1) & \psi_{12}(1) & \psi_{13}(1) & \psi_{14}(1) \\ \psi_{11}^{(2)}(1) & \psi_{12}^{(2)}(1) & \psi_{13}^{(2)}(1) & \psi_{14}^{(2)}(1) \end{pmatrix}. \tag{43b}$$

The equation

$$\Delta_1(A, \lambda) = 0 \tag{44}$$

is called the 'characteristic equation'† of the eigenvalue problem (19). For any fixed A the set of eigenvalues of (19) is given by the set of roots of the characteristic equation (44).

According to the properties of the functions $\psi_{1m}(\xi, A, \lambda)$ the determinant $\Delta_1(A, \lambda)$ is a regular analytic function of A and λ. In particular, for any given A the function $\Delta_1(A, \lambda)$ is a regular analytic function of λ. Because the solutions $\psi_{1m}(\xi, A, \lambda)$ obviously satisfy the relation

$$\psi_{1m}(\xi, A, \bar{\lambda}) = \overline{\psi_{1m}(\xi, A, \lambda)}; \qquad 0 \le \xi \le 1; \qquad A \ge 0; \qquad m = 1, 2, 3, 4 \tag{45}$$

the following equation holds for $\Delta_1(A, \lambda)$

$$\Delta_1(A, \bar{\lambda}) = \overline{\Delta_1(A, \lambda)}; \qquad (A \ge 0). \tag{46}$$

Therefore, if a non-real number $\lambda = \lambda_0$ is a root of the order k of equation (44) ($A \ge 0$), the same holds for $\bar{\lambda}_0$.

For any fixed $A (A \ge 0)$ there exist two possibilities

$$\text{(a)} \quad \Delta_1(A, \lambda) \not\equiv 0 \quad \text{in } \lambda; \qquad \text{(b)} \quad \Delta_1(A, \lambda) \equiv 0 \quad \text{in } \lambda. \tag{47}$$

According to the results of Section 2.1 case (b) is ruled out. Therefore $\Delta_1(A, \lambda)$ is a regular analytic function of λ, which is not identical to zero. Therefore for any $A \ge 0$ the equation (44) has at the most a countable number of zeros $\mu_n(A)$, $n = 1, 2, \ldots$ without finite accumulation points, i.e. the eigenvalue problem (19) possesses for any $A \ge 0$ at the most a countable number of eigenvalues $\lambda_1(A)$, $\lambda_2(A), \ldots$, which have no finite accumulation point. The notation $\lambda_n(A)$ and $\lambda_n^*(A)$, already introduced in Section 1, is thus justified. Later, in Section 2.5, we will decide how the enumeration of the eigenvalues should be achieved.

From (41) and (43b) one learns immediately that the rank of the matrix $D(A, \lambda)$ is never smaller than 2. Therefore the multiplicity of any eigenvalue $\lambda_n(A)$ which is given by the relation

$$[\text{multiplicity of the eigenvalue } \lambda_n(A)] = 4 - \text{rank } [D(A, \lambda_n(A))] \tag{48}$$

is never larger than 2.‡

† The notation 'secular equation' is also customary.

‡ An eigenvalue λ_n of the eigenvalue problem (19) is said to have the multiplicity $m (m > 0)$, if for $\lambda = \lambda_n$ the differential equation (19a) possesses exactly m linearly independent solutions which satisfy the boundary conditions (19b).

For $A = 0$ it can be proved easily that the eigenvalues $\lambda_n(0) = n^4\pi^4$, $n = 1, 2, \ldots$ [see equation (25)] have the multiplicity 1, see Section 2.4. For $A > 0$ the investigation of the multiplicity (which is always not larger than 2) is not that easy. We can omit it because we do not need the knowledge of these multiplicities for our purposes. In the following we will deal for a time rather with the roots $\mu_n(A)$ of the characteristic equation (44), taking into account their order instead of the eigenvalue $\lambda_n(A)$, realizing that any eigenvalue of the eigenvalue problem (19) is a root of the characteristic equation (44) and vice versa. In this way we avoid the difficulties originated by multiplicity of the eigenvalues, while we will be able to handle the order of the roots to the extent we need.

2.4. *Order of the roots of the characteristic equation* $\Delta_1(0, \lambda) = 0$

The eigenvalues of the eigenvalue problem (19) for A = 0 are given by $\lambda_n(0) = n^4\pi^4$, $n = 1, 2, \ldots$, see equation (25). Therefore the characteristic equation

$$\Delta_1(0, \lambda) = 0 \tag{49}$$

has exactly the roots $\lambda = n^4\pi^4$, $n = 1, 2, \ldots$. It is our next target to demonstrate that these roots are roots of order 1. In [18] this has been achieved by showing that

$$\left(\frac{\partial\Delta_1(0, \lambda)}{\partial\lambda}\right)_{\lambda = n^4\pi^4} \neq 0, \qquad n = 1, 2, \ldots. \tag{50}$$

Here we shall follow another path.

For $A = 0$ the eigenvalue problem (19) is self-adjoint. It is well known (see e.g. [19]) that for the self-adjoint problem the multiplicity of an eigenvalue $\lambda_n(0)$ equals the order of the root $\lambda = \lambda_n(0)$ of the characteristic equation. The multiplicity of the eigenvalues $\lambda_n(0) = n^4\pi^4$ is 1. This statement can easily be proved by employing the fundamental system

$$\begin{aligned} &\psi_{21}(\xi; n^4\pi^4) = e^{n\pi\xi}; \qquad &&\psi_{22}(\xi; n^4\pi^4) = e^{-n\pi\xi}; \\ &\psi_{23}(\xi; n^4\pi^4) = e^{in\pi\xi}; \qquad &&\psi_{24}(\xi; n^4\pi^4) = e^{-in\pi\xi}. \end{aligned} \tag{51}$$

Hence, the order of the roots $\lambda = n^4\pi^4$ of the characteristic equation is 1.

2.5. *The roots* $\mu_n(A)$ *of* $\Delta_1(A, \lambda) = 0$ *are continuous functions of* A

The determinant $\Delta_1(A, \lambda)$ is a regular analytic function of λ and A. As already has been proved in Section 2.3, there exist for all non-negative values of A at the most a countable number of roots $\mu_n(A)$ of the characteristic equation $\Delta_1(A, \lambda) = 0$, which have no finite accumulation point. Any root is of finite order. Let $\mathfrak{C}$ be a closed, finite loop in the complex λ-plane without double points, avoiding the zeros of $\Delta_1(A, \lambda)$. Let $N(A, \mathfrak{C})$ be the number of zeros $\mu_n(A)$, taking into account their order, which are encircled by $\mathfrak{C}$. Then, according to the residue theorem (see e.g. [20], Section 2.14), $N(A, \mathfrak{C})$ is given by the integral

$$N(A, \mathfrak{C}) = \frac{1}{2\pi i}\oint_{\mathfrak{C}} \frac{\partial\Delta_1(A, \lambda)}{\partial\lambda}\frac{d\lambda}{\Delta_1(A, \lambda)}. \tag{52}$$

Taking advantage of the continuity properties of the integrand and of the fact that the integral always equals a non-negative integer, one verifies easily (see again [20]) the following statement: Let $\mu_{n_0}(A)$ be a root of the order m of $\Delta(A_0, \lambda) = 0$, $A_0 \geq 0$. Then for any given $\varepsilon > 0$ there exist a circle $\mathfrak{C}[\varepsilon', \mu_{n_0}(A_0)]$ defined by

$$|\lambda - \mu_{n_0}(A_0)| \leq \varepsilon', \qquad 0 < \varepsilon' \leq \varepsilon \tag{53}$$

and a $\delta > 0$ in such a manner that the circle $\mathfrak{C}[\varepsilon', \mu_{n_0}(A_0)]$ encloses exactly m roots of $\Delta_1(A, \lambda) = 0$ for $A_0 - \delta < A < A_0 + \delta$, taking into account the order of the roots. (In the case $A_0 = 0$ we consider only the interval $A_0 \leq A < A_0 + \delta$.) It is superfluous to point out that the same statement holds if the point under consideration is not a root of $\Delta_1(A_0, \lambda) = 0$, i.e. if $m = 0$ holds.

In this way it has been demonstrated that the roots $\mu_n(A)$ depend continuously upon the parameter $A (A \geq 0)$, i.e. they form branches.

For the time being we have not introduced a 'global' enumeration of these branches. Let us first use a 'local' enumeration.

Let us agree that any root is evolving its own branch. So, if for $A = A_0 > 0$ the number λ_0 is a root of the order $m (m > 0)$ of $\Delta_1(A_0, \lambda) = 0$, we say that m branches coincide or intersect (as the case may be) for $A = A_0$ at λ_0. Then there exist for some $\delta > 0$ for all A in the interval $A_0 - \delta < A < A + \delta$ exactly m roots of $\Delta_1(A, \lambda) = 0$, say $\mu_{n_\sigma}(A)$, $\sigma = 1, \ldots, m$, with

$$\text{(a)} \quad \lim_{A \to A_0 - 0} \mu_{n_\sigma}(A) = \lambda_0, \qquad \text{(b)} \quad \lim_{A \to A_0 + 0} \mu_{n_\sigma}(A) = \lambda_0.\dagger \tag{54}$$

On the other hand, if for $\lambda = \lambda_0$ there exist exactly m branches with the property (54a) or (54b), then λ_0 is a root of the order m of the equation $\Delta_1(A_0, \lambda) = 0$. This proves that the number of 'incoming' branches equals the number of 'outgoing' branches, as A passes through A_0, and the intersection point λ_0 is a root of $\Delta_1(A_0, \lambda) = 0$, whose order is given by the number of intersecting branches. For $A_0 = 0$ we only consider right-side neighborhoods of A_0. The roots $\mu_n(0)$ of $\Delta_1(0, \lambda) = 0$ are already given by (25), any of these roots $\mu_n(0) = n^4\pi^4$, $n = 1, 2, \ldots$ is a root of the order 1, as has been demonstrated in Section 2.4. Exactly one branch rises from any of these points.

In order to introduce a 'global' enumeration of the roots, let us agree upon the following procedure. Starting from $A = 0$ we trace the roots, which is possible because of the continuity of the roots. That root of $\Delta_1(A, \lambda) = 0$, $(A > 0)$, which is located on that branch which coincides for $A = 0$ with $\lambda = \mu_n(0) = n^4\pi^4$ we call $\mu_n(A)$. No difficulties arise as long as no branches intersect each other or die out. As soon as for some $A_0 > 0$ two or more branches intersect, we have a certain freedom in numbering the involved branches for $A > A_0$. Here we make an arbitrary decision which, according to the conclusions outlined above, can be done in such a way that the involved branches are continued uninterruptedly. The number of branches which intersect for a certain A_0 at some point λ_0 can only be a finite one.

Several questions arise: (1) Can it happen that a certain branch, say $\mu_{n_0}(A)$, exists for $0 \leq A < A_0$, while it does not exist for $A = A_0$? (2) Do we obtain by this tracing procedure *all* the roots of $\Delta_1(A, \lambda) = 0$ for $A > 0$?

The first question can be answered in the negative by means of the following arguments: All the eigenvalues of (19), and therefore especially $\mu_{n_0}(A)$, are located in the union set, evolved by the circles (30). For $0 \leq A < A_0$ the root $\mu_{n_0}(A)$ is located in the

† $A \to A_0 - 0$ or $A \to A_0 + 0$ means that A is approaching A_0 from the left or from the right respectively.

union set evolved by the circles†

$$|\lambda - n^{*4}\pi^4| \leq A_0(n^*+1)\pi, \qquad n^* = 1, 2 \ldots. \tag{55}$$

For sufficiently large n^* these circles are separated. Because of (55) and because $\mu_{n_0}(A)$ is a continuous function of A, its absolute value is uniformly bounded for $0 \leq A < A_0$. Employing the Bolzano–Weierstrass accumulation point theorem and taking advantage of the continuity of the determinants $\Delta_1(A, \lambda)$ and of $\mu_{n_0}(A)$, one learns that for $A \to A_0 - 0$ the root $\mu_{n_0}(A)$ tends to a (finite) point λ_{n_0}, which is a root of $\Delta_1(A_0, \lambda) = 0$ too. So we obtain $\lambda_{n_0} = \mu_{n_0}(A_0)$. According to the results mentioned above the branch $\mu_{n_0}(A)$ exists even in a certain right-side neighborhood $A_0 \leq A < A_0 + \delta$ ($\delta > 0$) too.

In order to answer the second question in the affirmative, one has to rule out the possibility that a certain root of $\Delta_1(A, \lambda) = 0$, say $\hat{\mu}(A)$, exists for some right-side neighborhood $A_0 < A$ of a certain number $A_0 \geq 0$, while it ceases to exist for $A = A_0$. This possibility can be excluded by the arguments used for answering the first question, at which the number A_0 in the inequality (55) has to be replaced by $A_0 + 1$. So all the roots of $\Delta_1(A, \lambda) = 0$ are located on branches which can be traced backwards to $A = 0$, and for $A = 0$ all the roots are known, see (25). Therefore all roots (taking into account their order) of $\Delta_1(A, \lambda) = 0$, $A \geq 0$, are located on the aforementioned branches $\mu_n(A)$, which arise from the points $\mu_n(0) = n^4\pi^4$, $n = 1, 2, \ldots$.

Let us now investigate some necessary conditions in order that a certain root, say $\mu_{n_1}(A)$, can become complex. As has been mentioned in Section 2.3, non-real roots of (44) appear always in complex conjugate pairs. Let us assume that for a certain value $A_0 \geq 0$ the number $\mu_{n_1}(A_0)$ is a real root of $\Delta_1(A_0, \lambda) = 0$, while there exist for numbers A_σ with $A_\sigma \to A_0 + 0$ non-real roots $\mu_{n_1}(A_\sigma)$ of $\Delta_1(A_\sigma, \lambda) = 0$; then obviously the relation $\mu_{n_1}(A_\sigma) \to \mu_{n_1}(A_0)$ is valid. Then $\overline{\mu_{n_1}(A_\sigma)}$ is a root of $\Delta_1(A_\sigma, \lambda) = 0$ too with $\overline{\mu_{n_1}(A_\sigma)} \to \mu_{n_1}(A_0)$. Therefore $\mu_{n_1}(A_0)$ has to be at least a root of the order 2, i.e. $\mu_{n_1}(A)$ has to intersect or to coincide for $A = A_0$ with another branch.

For $A = 0$ it is $\mu_n(0) = n^4\pi^4$. Therefore, using the statements connected with the inequality (53), one realizes that for any n there exists an interval $0 \leq A \leq \delta_n (\delta_n > 0)$ in such a manner that for all A belonging to this interval the root $\mu_n(A)$ does not intersect with another root, and therefore it must be a real number in $0 \leq A \leq \delta_n$. Now, let us assume that (1) the root $\mu_{n_1}(A)$ is real in a certain interval $0 \leq A \leq A_{n_1} (A_{n_1} > 0)$, (2) there does not exist any root $\mu_n(A)$, which is real in a certain right-side neighborhood of A_{n_1} and tends to $\mu_{n_1}(A_{n_1})$ if $A \to A_{n_1} + 0$. (That means we assume that it is impossible to arrange by proper enumeration that $\mu_{n_1}(A)$ is real also in a certain interval $A_{n_1} < A < A_{n_1} + \delta$.) Then, because non-real roots always appear in pairs of conjugate complex numbers (see above), $\mu_{n_1}(A)$ must intersect for $A = A_{n_1}$ with another root $\mu_{n_2}(A)$ $(n_1 \neq n_2)$ which is real at least in a certain left-side neighborhood $A_{n_1} - \delta_{n_2} \leq A \leq A_{n_1} (\delta_{n_2} > 0)$ of A_{n_1}.

Now let us return to the eigenvalues. To introduce an enumeration of the eigenvalues we put

$$\lambda_n(A) = \mu_n(A), \qquad n = 1, 2, 3, \ldots \tag{56}$$

which is in agreement with the enumeration already imposed for $A = 0$ by (25). In the case where two or more branches $\mu_{n_1}(A), \ldots, \mu_{n_m}(A)$ coincide or intersect for a certain

† Again the author is indebted to Professor J. Moser for the idea to employ the results of Section 2.2 for dealing with these two questions.

number $A = A_0$ (then A_0 must be positive and the number of intersecting roots can only be a finite one!), we do not worry about the multiplicity of the eigenvalues $\lambda_{n_1}(A_0) = \ldots = \lambda_{n_m}(A_0)$; e.g. we will not worry if possibly there exist less than m linearly independent eigenfunctions for these m coinciding eigenvalues.

Using this enumeration, the eigenvalues $\lambda_n(A)$ are by virtue of the properties of the roots $\mu_n(A)$ continuous functions of $A, A \geq 0$. Especially the preceding statements concerning the reality of the $\mu_n(A)$ are valid for the $\lambda_n(A)$ too.

2.6. *The characteristic system*

The characteristic equation (44), which was constructed using the fundamental system (40), was very useful for gathering facts concerning the continuity of the eigenvalues, however it is not very suitable for the purpose of numerical calculations. So we establish in this section another characterization of the eigenvalues of (19), which is more appropriate for numerical investigations.

One manner in which a fundamental system of the differential equation (19a) may be found is the following. Substituting $f(\xi) = e^{-z\xi}$ into (19a) one obtains for z the 'auxiliary equation'†

$$p(z, A, \lambda) \equiv z^4 - Az - \lambda = 0. \qquad (A \geq 0). \tag{57}$$

This equation has four roots $z_m = z_m(A, \lambda)$; $m = 1, 2, 3, 4$. If these four roots are distinct, then the four functions

$$\psi_{3m}(\xi) = \psi_{3m}(\xi, A, \lambda) = e^{-z_m\xi}, \qquad m = 1, 2, 3, 4 \tag{58a}$$

form a fundamental system of (19a). If z_1 equals z_2, while z_3 and z_4 are distinct, then the fundamental system reads

$$\psi_{31}(\xi) = e^{-z_1\xi}; \qquad \psi_{32}(\xi) = \xi e^{-z_1\xi}; \qquad \psi_{33}(\xi) = e^{-z_3\xi}; \qquad \psi_{34}(\xi) = e^{-z_4\xi} \tag{58b}$$

and so on, see e.g. [17]. For $\lambda \neq 0$ the auxiliary equation (57) has roots at most of the order 2, as a simple calculation proves.

Using this fundamental system $\psi_{3m}(\xi, A, \lambda)$, $m = 1, 2, 3, 4$, one proves that a certain number $\lambda = \lambda_0$ is an eigenvalue of the eigenvalue problem (19) if and only if the four roots $z_m(A, \lambda_0)$ of equation (57) for $\lambda = \lambda_0$ satisfy the equation

$$\Delta_3(z_1, z_2, z_3, z_4) = 0, \tag{59a}$$

where Δ_3 is defined by

$$\Delta_3(z_1, z_2, z_3, z_4) \equiv \frac{1}{g(z_1, z_2, z_3, z_4)} \begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{-z_1} & e^{-z_2} & e^{-z_3} & e^{-z_4} \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 \\ z_1^2 e^{-z_1} & z_2^2 e^{-z_2} & z_3^2 e^{-z_3} & z_4^2 e^{-z_4} \end{vmatrix} \tag{59b}$$

with

$$g(z_1, z_2, z_3, z_4) \equiv (z_1 - z_2)(z_1 - z_3)(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_3 - z_4). \tag{59c}$$

† This equation is sometimes also called the 'characteristic equation,' but in this paper this notation will be strictly reserved for equation (44) which defines the eigenvalues of (19).

This is obvious so long as for $\lambda = \lambda_0$ the four roots of (57) are distinct, then the factor $[g(z_1, z_2, z_3, z_4)]^{-1}$ is meaningless. But also in the case where there are roots of higher order than 1 of (57) the boundary conditions (19b) lead by virtue of the factor $[g(z_1, z_2, z_3, z_4)]^{-1}$ to the condition (59a) as one can verify by straightforward calculations.

In order to transform these conditions in a more convenient form, let us first gather some simple properties of the roots of the auxiliary equation (57), which can be confirmed easily. (1) If λ is real and positive (all real eigenvalues of (19) are positive, see Section 2.1), then equation (57) has two real roots (one positive, one negative) and one pair of (non-real) conjugate complex roots. (2) If λ is non-real, then there exist no real roots and no pairs of conjugate complex roots. (3) For $A = 0$ and $\lambda \neq 0$ the four roots are distinct. (4) The four roots $z_m(A, \lambda)$ depend continuously upon the parameters A and λ. (This is a well-known fact which can be verified by arguments analogous to those already used in Section 2.5.) (5) From the coefficients of the polynomial $p(z, A, \lambda)$ one obtains immediately the following relations:

$$\text{(a)}\quad z_1+z_2+z_3+z_4 = 0; \qquad \text{(b)}\quad z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+z_3z_4 = 0;$$
$$\text{(c)}\quad z_1z_2z_3+z_1z_2z_4+z_1z_3z_4+z_2z_3z_4 = A; \qquad \text{(d)}\quad z_1z_2z_3z_4 = -\lambda. \tag{60}$$

The first two relations lead to the equations

$$z_3+z_4 = -(z_1+z_2); \qquad z_3z_4 = -z_1z_2+(z_1+z_2)^2 \tag{61}$$

from which one obtains

$$z_{3,4} = -\tfrac{1}{2}(z_1+z_2)\pm\sqrt{[z_1z_2-\tfrac{3}{4}(z_1+z_2)^2]}. \tag{62}$$

Therefore, as soon as two roots of (57) are known, the two remaining roots are known by virtue of (62). Until now no decisions have been made on how to number the roots z_m. This will be done later.

Now we take two roots, z_1 and z_2, of (57) and introduce two new parameters α and β by letting†

$$\alpha = \frac{1}{2}(z_1+z_2); \qquad \beta = \frac{1}{2i}(z_1-z_2) \tag{63a}$$

and obtain

$$z_1 = \alpha+i\beta; \qquad z_2 = \alpha-i\beta. \tag{63b}$$

Then, by virtue of equation (62) we arrive at

$$z_3 = -\alpha+\sqrt{(\beta^2-2\alpha^2)}; \qquad z_4 = -\alpha-\sqrt{(\beta^2-2\alpha^2)}. \tag{63c}$$

Further, equations (60c) and (60d) lead to

$$\text{(a)}\quad A = 4\alpha(\alpha^2-\beta^2); \qquad \text{(b)}\quad \lambda = (\alpha^2+\beta^2)(\beta^2-3\alpha^2). \tag{64}$$

From (59b) and (59c) one obtains

$$g(z_1, z_2, z_3, z_4) \equiv g(\alpha, \beta) = 16i\beta\sqrt{(\beta^2-2\alpha^2)}[(3\alpha^2-\beta^2)^2+4\alpha^2\beta^2] \tag{65}$$

† This device has already been used by A. A. Movchan [5].

and†

$$\Delta_3(z_1, z_2, z_3, z_4) \equiv F(\alpha, \beta)$$

$$= \frac{(3\alpha^4+\beta^4-2\alpha^2\beta^2)\dfrac{\sin\beta}{\beta}\cdot\dfrac{\sinh\sqrt{(\beta^2-2\alpha^2)}}{\sqrt{(\beta^2-2\alpha^2)}}-2\alpha^2\cos\beta\cosh\sqrt{(\beta^2-2\alpha^2)}+2\alpha^2\cosh(2\alpha)}{(3\alpha^2-\beta^2)^2+4\alpha^2\beta^2} \tag{66}$$

So we face the following situation: Let λ_0 be an eigenvalue of (19) for a fixed $A \geq 0$. Then the auxiliary equation (57) possesses for $\lambda = \lambda_0$ four roots z_m which satisfy equations (59a) and (60) with $\lambda = \lambda_0$. The corresponding numbers α and β evolved by these roots according to (63a) satisfy the equations

$$\text{(a)}\quad F(\alpha, \beta) = 0; \qquad \text{(b)}\quad A = 4\alpha(\alpha^2-\beta^2) \tag{67}$$

and (64b) for $\lambda = \lambda_0$. On the other hand, for any pair of numbers α, β, which satisfy for a given $A \geq 0$ equations (67), there exist according to equations (63) and (64b) four numbers z_m and a number $\lambda = \lambda_0$. These numbers z_m and λ_0 satisfy equations (60), the z_m are therefore roots of equation (57) for $\lambda = \lambda_0$ and for the value A under consideration. In addition, these four numbers z_m satisfy equation (59a). Therefore λ_0 is an eigenvalue of (19) for the considered value of A.

Therefore one obtains the set of eigenvalues of (19) for any $A \geq 0$ in the following way: One determines all simultaneous solutions α, β of equations (67), and obtains by substitution of these solutions α, β into (64b) all eigenvalues for the parameter value A under consideration. Therefore the two equations (67) are called the 'characteristic system'.

Eliminating β by using (67b), one can replace equations (67) for $A > 0$ by

$$F^*(A, \alpha) = \frac{1}{7\alpha^4+\left(\alpha^2-\dfrac{A}{4\alpha}\right)^2}\left[\left(2\alpha^4+\frac{A^2}{16\alpha^2}\right)\frac{\sin\sqrt{\left(\alpha^2-\dfrac{A}{4\alpha}\right)}}{\sqrt{\left(\alpha^2-\dfrac{A}{4\alpha}\right)}}\,\frac{\sin\sqrt{\left(\alpha^2+\dfrac{A}{4\alpha}\right)}}{\sqrt{\left(\alpha^2+\dfrac{A}{4\alpha}\right)}}\right.$$
$$\left.-2\alpha^2\cos\sqrt{\left(\alpha^2-\frac{A}{4\alpha}\right)}\cos\sqrt{\left(\alpha^2+\frac{A}{4\alpha}\right)}+2\alpha^2\cosh(2\alpha)\right] = 0. \tag{68}$$

Then λ is given in terms of α and A by

$$\lambda = \frac{A^2}{16\alpha^2}-4\alpha^4 \tag{69}$$

for $A > 0$, which results in $\alpha \neq 0$, see equation (67b).

Both forms (67) and (68) of the characteristic system will be used in the following. One observes that $F(\alpha, \beta)$ is an even function in α and β. Obviously one has not to worry about the question of which of the two possible square roots should be used in (67) and (68).

It is obvious that, in general, the characteristic system (67) has twelve solutions α, β which lead to the same eigenvalue. This is due to the fact that, in general, there are twelve possible ways of deciding which of the four roots of the auxiliary equation (57)

† One obtains this result also by putting $k = 0$ in Movchan's equation (2.2) in [6].

should be the pair z_1, z_2. It is sufficient for us to know one of these twelve solutions. In order to achieve this we introduce a 'global' enumeration of the four roots z_m. We consider the auxiliary equation

$$z^4 - Az - \lambda_n(A) = 0, \qquad A \geq 0, \qquad n = 1, 2, \ldots \tag{70}$$

for increasing A, starting with $A = 0$. $\lambda_n(A)$ is the nth eigenvalue of the eigenvalue problem (19) where the index n is fixed by (56). The eigenvalues $\lambda_n(A)$ are continuous functions of A, see Section 2.5. Therefore [see the statements preceding equation (60)] the four roots $z_{nm}(A)$, $m = 1, 2, 3, 4$, of (70) are continuous functions of A too. For $A = 0$ it is $\lambda_n(0) = n^4\pi^4$, then the four roots of (70) are the numbers

$$n\pi, \qquad -n\pi, \qquad in\pi, \qquad -in\pi. \tag{71}$$

Now we put

$$z_{n1}(0) = in\pi; \qquad z_{n2}(0) = -in\pi; \qquad z_{n3}(0) = n\pi; \qquad z_{n4}(0) = -n\pi \tag{72}$$

and we agree that for $A > 0$ the index m of the root $z_{nm}(A)$, $m = 1, 2, 3, 4$ is arranged in such a way that $z_{nm}(A)$ is located on that branch which coincides for $A = 0$ with $z_{nm}(0)$. In case of intersections for $A > 0$ we make an arbitrary decision on the numeration of these branches after this coincidence took place in such a way that the branches $z_{nm}(A)$ are continued uninterruptedly. Having settled this, one realizes that the numbers

$$\alpha_n(A) = \frac{1}{2}[z_{n1}(A) + z_{n2}(A)]; \qquad \beta_n(A) = \frac{1}{2i}[z_{n1}(A) - z_{n2}(A)];$$
$$n = 1, 2, \ldots; \qquad A \geq 0 \tag{73}$$

are continuous functions of A too, and it holds, in particular

$$\alpha_n(0) = 0; \qquad \beta_n(0) = n\pi; \qquad n = 1, 2, \ldots. \tag{74}$$

The important point is that the $\alpha_n(A)$ and $\beta_n(A)$ can be traced as continuous functions of $A (A \geq 0)$, starting from $A = 0$ at the points given by (74).

3. THE EIGENVALUES FOR $0 \leq A \leq 110{,}000$

In this section we will obtain a survey of the eigenvalues $\lambda_n(A)$ of the eigenvalue problem (19) to the extent we need. We will restrict ourselves to the interval $0 \leq A \leq 110{,}000$.

In order to find the critical speeds of the flutter problem under investigation, we have to proceed afterwards in the following way. Having fixed all parameters besides M—the parameter A is proportional to M—we have to find the critical value of A, A_{crit}, with the properties listed in (24).

Since only non-real eigenvalues can be located on the outside of the stability parabola [see (29)], we have to focus our attention mainly upon the non-real eigenvalues of (19). There does not exist any theorem which insures that the first two eigenvalues $\lambda_1(A)$ and $\lambda_2(A)$ are the first ones to cross the stability parabola as A is increasing. Therefore, A_{crit} can be determined only after all eigenvalues, which are probably non real in $0 \leq A \leq A_{\text{crit}}$, are taken into account.

All eigenvalues $\lambda_n(A)$ and therefore [see the statement (1), preceding equation (60)] all $\alpha_n(A)$ and $\beta_n(A)$ are real numbers in certain (probably individual) right-side neighborhoods

$0 \leq A \leq \delta_n$ of $A = 0$. Therefore, we will first investigate the characteristic system (67) and equation (64) for real α and β. Later we will proceed to the investigation of the complex eigenvalues, using equations (68) and (69).

In order to examine the reliability of Galerkin's method, the exact results for the eigenvalues with those obtained by Galerkin's method will be compared at the end of this section.

3.1. *The characteristic system for real α and β*

Let us first investigate the curves $\mathfrak{C}_A$, defined by the equation $A = 4\alpha(\alpha^2 - \beta^2)$ [see (67)] for $A \geq 0$ in the real $\alpha\beta$-plane†. First of all one notices that these curves are symmetric with respect to the α-axis. For $A = 0$ the curve $\mathfrak{C}_0$ consists of the three lines

$$\alpha = 0; \qquad \beta = +\alpha; \qquad \beta = -\alpha. \tag{75}$$

For $A > 0$ the curves $\mathfrak{C}_A$ are split up into three parts (see Fig. 2); one part, $\mathfrak{C}_A^+$, for $\alpha > 0$

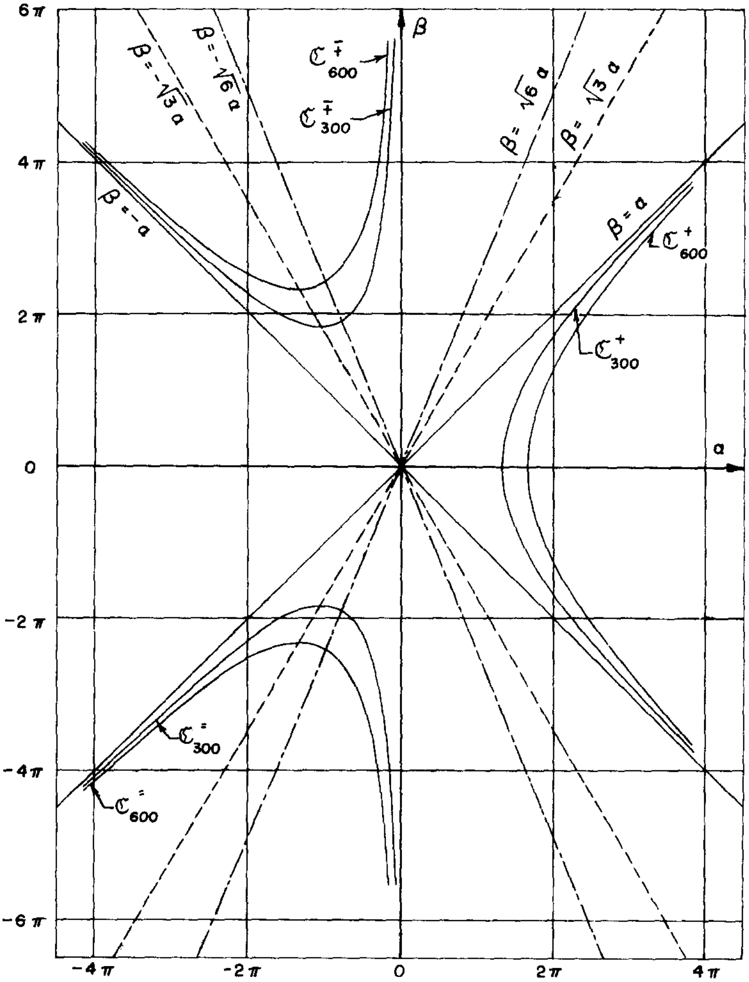

FIG. 2. The curves $\mathfrak{C}_A$, defined by $A = 4\alpha(\alpha^2 - \beta^2)$.

† The curves $\mathfrak{C}_A$ have already been discussed by Movchan [5] in Section 2.

and two parts, $\mathfrak{C}_A^{\mp}$ and $\mathfrak{C}_A^{=}$ for $\alpha < 0$. The points on $\mathfrak{C}_A^{+}$ satisfy the inequality $|\beta| < \alpha$. The curves $\mathfrak{C}_A^{+}$ are of minor interest, because they will not lead to real solutions of the characteristic system (67), as will be shown below. The points on the parts $\mathfrak{C}_A^{\mp}$ and $\mathfrak{C}_A^{=}$ satisfy the inequality $|\beta| > -\alpha$. One of the two parts, $\mathfrak{C}_A^{\mp}$, takes its course in the upper $\alpha\beta$-plane; the other one, $\mathfrak{C}_A^{=}$, takes its course in the lower $\alpha\beta$-plane. One obtains $\mathfrak{C}_A^{=}$ by reflecting $\mathfrak{C}_A^{\mp}$ at the α-axis. Therefore, it is sufficient to deal only with $\mathfrak{C}_A^{\mp}$. The curves $\mathfrak{C}_A^{\mp}$ have a hyperbolic-like shape and the lines $\alpha = 0$ and $\beta = -\alpha$ are asymptotic to them. The minimum of $\mathfrak{C}_A^{\mp}$ is located on the line $\beta = -\sqrt{(3)}\alpha$. The minimum of $\mathfrak{C}_A^{\mp}$ becomes smaller for decreasing A and tends to zero as A tends to zero. The curve $\mathfrak{C}_{A_1}^{\mp}$ is located in the inside of $\mathfrak{C}_{A_2}^{\mp}$ if $A_2 < A_1$ holds.

Now we consider† the behavior of the numbers [see (64)] $\lambda = (\alpha^2+\beta^2)(\beta^2-3\alpha^2)$ if the point (α, β) moves along $\mathfrak{C}_A^{\mp}$ for a fixed A. One obtains $(\alpha < 0)$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\alpha} = \frac{\partial\lambda}{\partial\alpha} + \frac{\partial\lambda}{\partial\beta}\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} = -\frac{2}{\alpha}[(3\alpha^2-\beta^2)^2+4\alpha^2\beta^2] > 0. \tag{76}$$

Therefore, λ is increasing as the point (α, β) moves along $\mathfrak{C}_A^{\mp}$ from the left to the right.

Now we investigate the function $F(\alpha, \beta)$, defined by the equation (66), in the real $\alpha\beta$-plane. After some thorough calculations, one can show that the equation $F(\alpha, \beta) = 0$ [see (67a)] defines closed loops l_{2m}, $m = \pm 1, \pm 2, \ldots$, of which some are shown in Fig. 3.

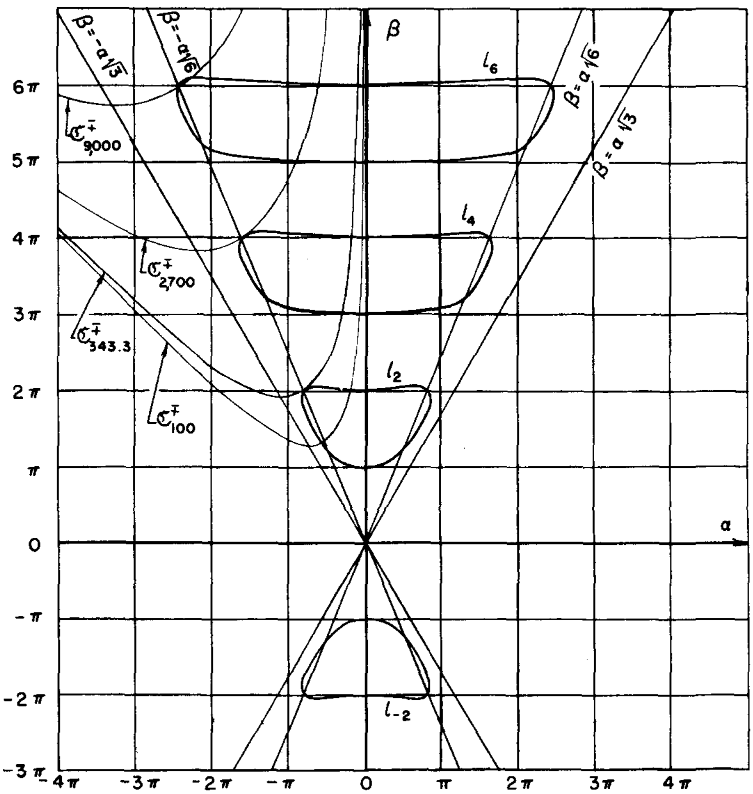

FIG. 3. The loops l_{2m}, defined by $F(\alpha, \beta) = 0$.

† See also Section 2 of [5]. There the partial differential quotient has to be replaced by the total one.

The loops $l_2, l_4, \ldots$ are located in the upper $\alpha\beta$-half-plane, and the loops $l_{-2}, l_{-4}, \ldots$ are located in the lower $\alpha\beta$-half-plane. One obtains the loop l_{-2m} by reflecting l_{2m} on the α-axis. The loops are symmetric with respect to the β-axis. The loops l_{2m}, $m = 1, 2, \ldots$ have, in particular, the following properties:

(1) l_{2m} passes through the points

$$(0, (2m-1)\pi), (0, 2m\pi), \quad \left(\frac{2m\pi}{\sqrt{6}}, 2m\pi\right), \quad \left(-\frac{2m\pi}{\sqrt{6}}, 2m\pi\right).$$

(2) At the points $(0, (2m-1)\pi)$ and $(0, 2m\pi)$ the loop l_{2m} has a relative minimum. (77)

(3) For the points (α, β) on l_{2m} the following inequalities are valid:

$$(2m-1)\pi \leq \beta < (2m+1)\pi;$$

$$-\frac{\beta}{\sqrt{3}} < \alpha < \frac{\beta}{\sqrt{3}}; \qquad (2m-1)\pi < \beta \qquad \text{for} \qquad \alpha \neq 0.$$

All the loops l_{2m} have a similar shape. According to the inequalities listed under (77), the loops l_{2m} do not intersect with the curves $\mathfrak{C}_A^+$.

Now the intersections of the curves $\mathfrak{C}_A$ with the loops l_{2m} will be investigated. The coordinates of these intersections are real solutions of the characteristic system (67) and lead, according to (64b), to real eigenvalues of (19).

For $A = 0$ the curve $\mathfrak{C}_0$ intersects the loop l_{2m}, $m = 1, 2, \ldots$ at the points

$$(\alpha^2{}_{m-1} = 0; \beta_{2m-1} = (2m-1)\pi), \qquad (\alpha_{2m} = 0; \beta_{2m} = 2m\pi), \qquad m = 1, 2, \ldots. \tag{78}$$

The coordinates α and β of these points are the solutions of the characteristic system (67), listed previously under (74). The eigenvalues, which belong to the intersections (78) are, according to (64b)

$$\lambda_{2m-1}(0) = (2m-1)^4\pi^4; \qquad \lambda_{2m}(0) = (2m)^4\pi^4; \qquad m = 1, 2, \ldots. \tag{79}$$

We can neglect the intersections of $\mathfrak{C}_0$ with the loops $l_{-2}, l_{-4}, \ldots$, because any of these intersections gives one of the eleven superfluous solutions of the characteristic system (67), which originates from an interchange of the roots z_1 and z_2 of the auxiliary equation (57).

For $A > 0$ the curve $\mathfrak{C}_A^+$ has no intersections with the loops l_{2m}, as already has been mentioned. Because of the symmetry with respect to the α-axis we can restrict ourselves to the investigation of the intersections of the curves $\mathfrak{C}_A^\mp$ with the loops l_{2m}, $m = 1, 2, \ldots$, employing the same arguments as above.

Taking advantage of the features of the loops l_{2m} and the curves $\mathfrak{C}_A^\mp$, already mentioned above, one obtains: For a certain value of A, which we denote by $\mathring{A}_{2m}$, the curve $\mathfrak{C}_A^\mp$ touches the loop l_{2m} at one point. For $0 < A < \mathring{A}_{2m}$ the curve $\mathfrak{C}_A^\mp$ has two (distinct) intersections with the loop l_{2m}, for $A > \mathring{A}_{2m}$ there are no longer intersections between $\mathfrak{C}_A^\mp$ and l_{2m}. For $0 < A \leq \mathring{A}_{2m}$ the curve $\mathfrak{C}_A^\mp$ has two distinct intersections with any of the loops $l_{2m+2}, l_{2m+4}, \ldots$. For $A = A_{2m}^* = 80m^3\pi^3/(3\sqrt{6})$ the curve $\mathfrak{C}_A^\mp$ passes through the point $(-2m\pi/\sqrt{6}, 2m\pi)$, which is located on the loop l_{2m}, see (77). Therefore we find

$$A_{2m}^* \equiv \frac{80m^3\pi^3}{3\sqrt{6}} \leq \mathring{A}_{2m}, \qquad m = 1, 2, \ldots. \tag{80}$$

By some more careful calculations, using e.g. graphical methods, one obtains (these are approximate values)

$$\mathring{A}_2 = 343{\cdot}3; \qquad \mathring{A}_4 = 2{,}726; \qquad \mathring{A}_6 = 9{,}185; \tag{81}$$

while for the values on the left-hand side of the inequality (80) one obtains

$$\begin{aligned} A_2^* &= 337{\cdot}6; & A_4^* &= 2{,}700; & A_6^* &= 9{,}114; & A_8^* &= 21{,}603; \\ A_{10}^* &= 42{,}194; & A_{12}^* &= 72{,}912; & A_{14}^* &= 115{,}780; & A_{16}^* &= 172{,}830. \end{aligned} \tag{82}$$

The two intersections of the curve $\mathfrak{C}_A^{\mp}$ and the loop l_{2m} in $0 < A < \mathring{A}_{2m}$ tend for $A \to +0$ to the corresponding points listed under (78). Therefore we call them, in agreement with the notations already introduced in Section 2.6

$$(\alpha_{2m-1}(A), \beta_{2m-1}(A)); \qquad (\alpha_{2m}(A), \beta_{2m}(A)); \; 0 < A \leq \mathring{A}_{2m}, \qquad m = 1, 2, \ldots, \tag{83}$$

and it holds for $A \to +0$

$$\begin{aligned} \alpha_{2m-1}(A) &\to 0; & \beta_{2m-1}(A) &\to (2m-1)\pi; \\ \alpha_{2m}(A) &\to 0; & \beta_{2m}(A) &\to 2m\pi. \end{aligned} \tag{84}$$

For $A = \mathring{A}_{2m}$ we call the point of contact

$$(\alpha_{2m-1}(\mathring{A}_{2m}), \beta_{2m-1}(\mathring{A}_{2m})) = (\alpha_{2m}(\mathring{A}_{2m}), \beta_{2m}(\mathring{A}_{2m})), \qquad m = 1, 2, \ldots. \tag{85}$$

The eigenvalues, belonging according to (64b) to these intersections, are real and because of (76) distinct; by virtue of (84) they tend for $A \to +0$ to the eigenvalues

$$\lambda_{2m-1}(0) = (2m-1)^4\pi^4 \qquad \text{and} \qquad \lambda_{2m}(0) = (2m)^4\pi^4,$$

respectively. Therefore, according to the enumeration introduced in Section 2.5, they are the eigenvalues $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$. Taking advantage again of the relation (76) one learns

$$\lambda_n(A) < \lambda_{n+1}(A) \quad \text{for} \quad 0 \leq A < \mathring{A}_2, \qquad n = 1, 2, \ldots. \tag{86}$$

The eigenvalues $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ are determined by these intersections for $0 < A < \mathring{A}_{2m}$ as real numbers, and it holds

$$\lambda_{2m-1}(A) < \lambda_{2m}(A) < \lambda_{2m+1}(A) < \ldots \quad \text{for} \quad 0 \leq A < \mathring{A}_{2m}. \tag{87}$$

For $A = \mathring{A}_{2m}$ the eigenvalues $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ coincide, and one obtains

$$\lambda_{2m-1}(A) = \lambda_{2m}(A) < \lambda_{2m+1}(A) < \ldots \qquad \text{for} \qquad A = \mathring{A}_{2m}, \qquad m = 1, 2, \ldots. \tag{88}$$

Some of these results, which were obtained by graphical methods, are shown in Figs. 4 and 5.

What happens to $\lambda_1(A)$ and $\lambda_2(A)$ if A passes increasingly through $\mathring{A}_2$? Because of $\lambda_1(\mathring{A}_2) = \lambda_2(\mathring{A}_2)$, the necessary conditions, mentioned in Section 2.5, are satisfied in order that $\lambda_1(A)$ and $\lambda_2(A)$ are non-real for larger A. We know that $\alpha_n(A)$ and $\beta_n(A)$, $n = 1, 2$ proceed continuously as A increases. As the curves $\mathfrak{C}_A^{\mp}$ have no (real) intersections with l_2 for $A > \mathring{A}_2$, there exists at least a certain open interval $\mathring{A}_2 < A < \mathring{A}_2 + \delta$, $\delta > 0$, where not both of the functions $\alpha_1(A)$ and $\beta_1(A)$ are simultaneously real numbers; the same applies to $\alpha_2(A)$ and $\beta_2(A)$. Then the roots $z_{11}(A)$ and $z_{12}(A)$ of the auxiliary equation (70) for $n = 1$ are for $\mathring{A}_2 < A < A + \delta$ no longer a pair of conjugate complex numbers, see (73), but they are in consequence of their continuity non-real numbers at

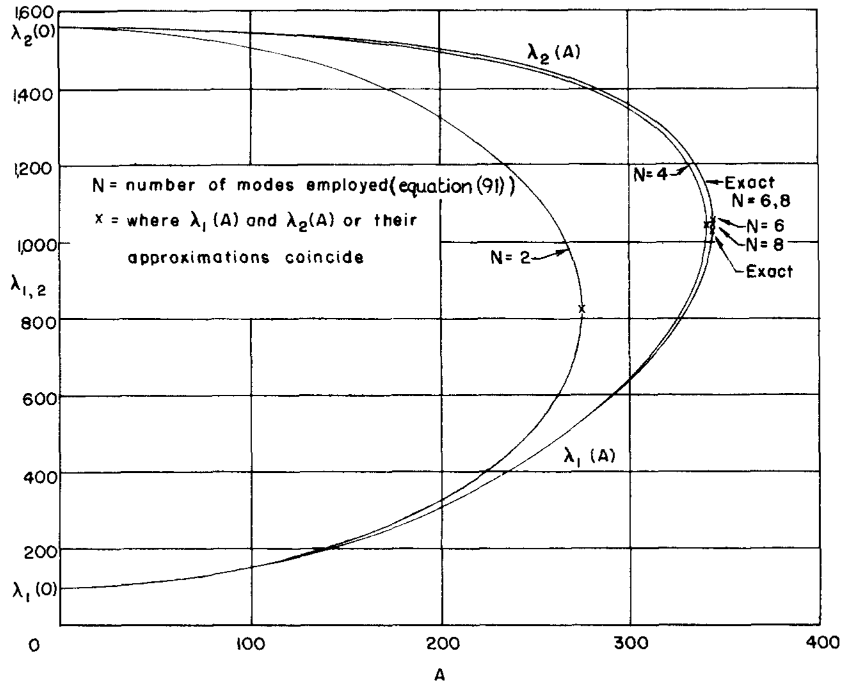

FIG. 4. The eigenvalues $\lambda_1(A)$ and $\lambda_2(A)$ for $0 \leq A \leq \mathring{A}_2$. Comparison of the exact results with those obtained by Galerkin's method.

least in some interval $\mathring{A}_2 \leq A < \mathring{A}_2 + \delta'$, $\delta \geq \delta' > 0$. These circumstances rule out the possibility that $\lambda_1(A)$ is a real number for $\mathring{A}_2 < A < \mathring{A}_2 + \delta'$, see the statements preceding equations (60). Because non-real eigenvalues appear always in complex conjugate pairs, we arrive at $\lambda_1(A) = \overline{\lambda_2(A)} \neq$ real number at least for $\mathring{A}_2 < A < \mathring{A}_2 + \delta'$. Later it will be found that $\lambda_1(A)$ and $\lambda_2(A)$ evolve a pair of conjugate complex eigenvalues at least in the interval $\mathring{A}_2 < A \leq 110{,}000$.

In general, the eigenvalues $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ are real and distinct for $0 \leq A < \mathring{A}_{2m}$. They coincide for $A = \mathring{A}_{2m}$ and subsequently become conjugate complex. This can be confirmed by employing the same arguments as above.

In this way one already knows, for any $A \geq 0$, which of the eigenvalues $\lambda_n(A)$ are probably complex numbers before any determination of complex eigenvalues has been carried out. In this connection the inequality (80) and the list (82) are illuminating.

3.2. *Tracing of complex eigenvalues*

As soon as A becomes larger than $\mathring{A}_{2m}$, the eigenvalues $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ can no longer be determined by the graphical method mentioned in Section 3.1. For $A > \mathring{A}_{2m}$ equation (68) has been used for the determination of $\alpha_{2m-1}(A)$ and $\alpha_{2m}(A)$. This was done by employing the 'method of false position', increasing A step-by-step and always using the result of the last step as the first guess for the next step. Then $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ were found by equation (69). As long as $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ are non-real, the relation

$$\lambda_{2m-1}(A) = \overline{\lambda_{2m}(A)}; \qquad m = 1, 2, \ldots \tag{89}$$

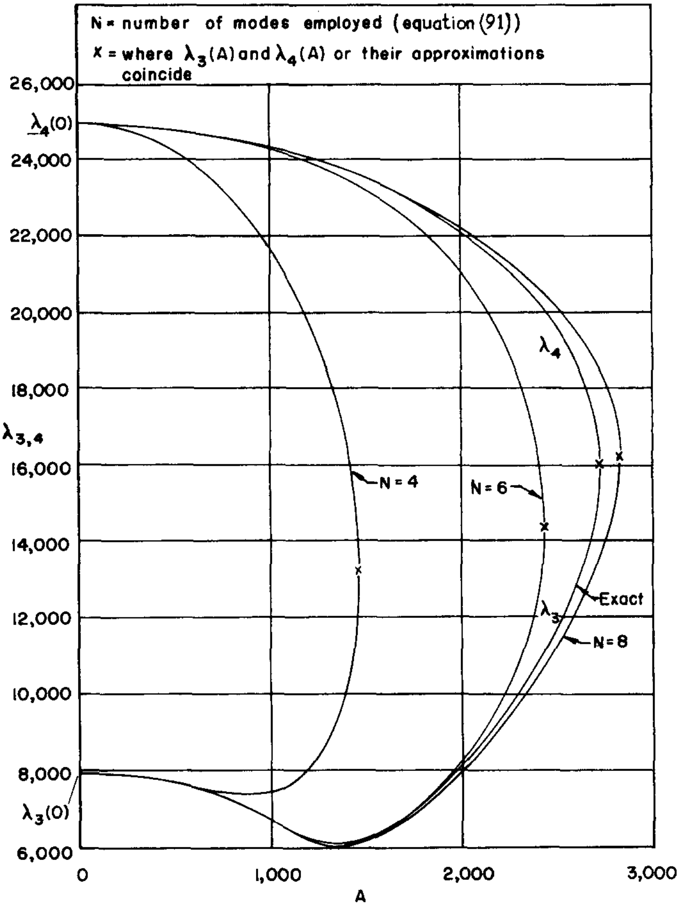

FIG. 5. The eigenvalues $\lambda_3(A)$ and $\lambda_4(A)$ for $0 \leq A \leq \mathring{A}_4$. Comparison of the exact results with those obtained by Galerkin's method.

holds. An arbitrary choice in numbering these two eigenvalues has to be made as soon as A passes through $\mathring{A}_{2m}$, as mentioned in Section 2.5. The decision was made that $\lambda_{2m-1}(A)$ should be that eigenvalue with the positive imaginary part,

$$\operatorname{Im}[\lambda_{2m-1}(A)] > 0; \qquad m = 1, 2, \ldots, \tag{90}$$

after A passes through $\mathring{A}_{2m}$. No further decisions were needed, because in the investigated intervals $\mathring{A}_{2m} < A \leq 110{,}000$ the eigenvalues $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$, $m = 1, 2, \ldots, 6$ stayed non-real and separated from each other, while the eigenvalues $\lambda_n(A)$, $n = 13, 14, \ldots$ are still real and separated in $0 \leq A \leq 110{,}000$, as one learns from the inequality (80) and the list (82). The calculations were done on a Burroughs 220 computer, available at the California Institute of Technology. The results are shown in Figs. 6 and 7. (The broken lines in Fig. 7 connect eigenvalues which belong to the same A-value respectively.)

One basic feature of the eigenvalues $\lambda_n(A)$ for $0 \leq A \leq 110{,}000$ should be pointed out. As has been mentioned in the beginning of Section 3, we are mainly concerned with the question of whether or not for a certain $A > 0$ there are already any eigenvalues $\lambda_n(A)$

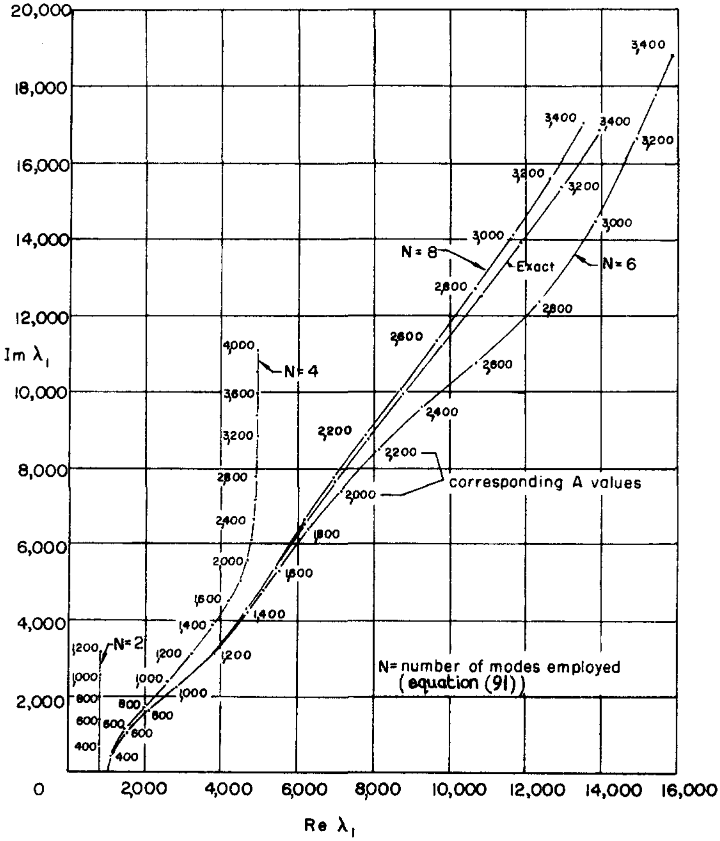

FIG. 6. The eigenvalue $\lambda_1(A)$ for $\mathring{A} \leq A \leq 3400$. Comparison of the exact results with those obtained by Galerkin's method.

located on the outside of the stability parabola (23b). Since all real eigenvalues are positive and therefore on the inside of the parabola, only non-real eigenvalues are to be taken into account. From the curves $\lambda_n(A)$ in Fig. 7, one can draw immediately the following conclusion. If for a certain interval $0 \leq A \leq A_0$ ($A_0 \leq 110{,}000$) the eigenvalue $\lambda_1(A)$ is on the inside or on the boundary of the stability parabola, then all eigenvalues $\lambda_3(A)$, $\lambda_4(A), \ldots$ are on the inside of this parabola for $0 \leq A \leq A_0$. Therefore, for the purposes of the flutter problem under investigation and for gathering the results in Section 4, one only has to deal with the eigenvalue $\lambda_1(A)$. [For $A > \mathring{A}_2$ it is $\lambda_1(A) = \overline{\lambda_2(A)}$.]

3.3. *Comparison of the exact eigenvalues with those obtained by Galerkin's method*

One of the purposes of this investigation is to check whether or not Galerkin's method leads to reliable results if applied to the eigenvalue problem under consideration. The reliability of Galerkin's method has been proved only for a certain class of self-adjoint eigenvalue problems (see e.g. [21]). But in spite of this fact, Galerkin's method is used as a handy tool for nearly all eigenvalue problems occurring in engineering. However, there are cases of non-self-adjoint eigenvalue problems known—e.g. the eigenvalue

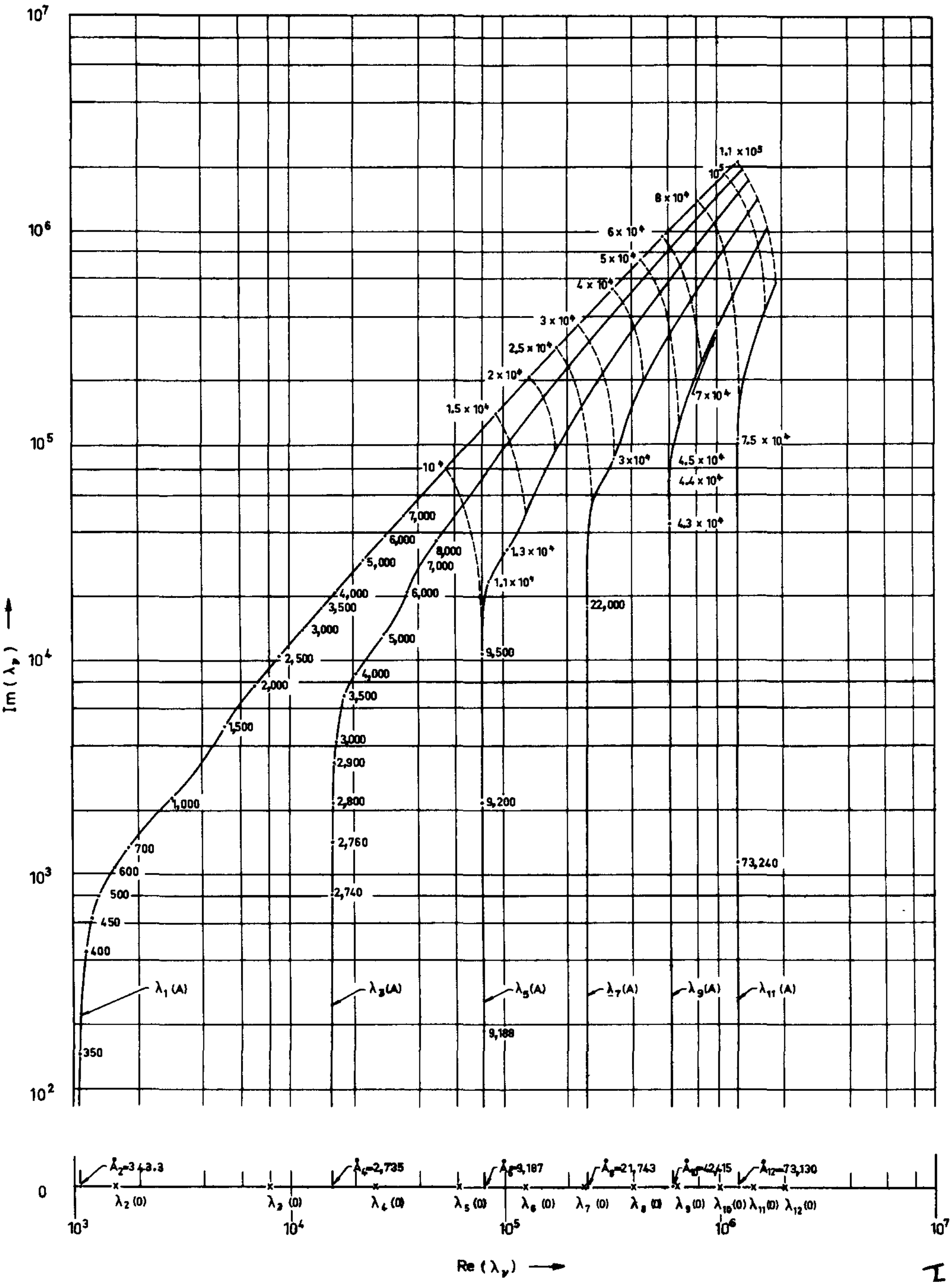

FIG. 7. The eigenvalue $\lambda_{2m-1}(A)$ for $\dot{A}_{2m} \leq A \leq 110{,}000$, $m = 1, 2, 3, 4, 5, 6$. The dots are labeled with the corresponding A-values.

problem connected with the flutter of a panel without bending rigidity (see [1], Section 2.1)—where Galerkin's method led to spurious eigenvalues.

For the eigenvalue problem (19) Galerkin's method has been used employing 4, 6 and 8 modes respectively. The comparison functions were of the form

$$\mathring{w}(\xi) = \sum_{n=1}^{N} c_n \sin(n\pi\xi); \qquad N = 4, 6, 8. \tag{91}$$

Again, the calculations were carried out on the Burroughs 220 computer. Some of the results are shown in Figs. 4, 5, and 6 together with the exact results. These curves demonstrate that Galerkin's method leads to dependable results in applying it to the non-self-adjoint eigenvalue problem (19). However, the larger the value of A, the more modes one has to employ to obtain reliable results.

From the results shown in Figs. 4 and 5 one learns the following interesting fact: As long as an eigenvalue $\lambda_n(A)$ of the non-self-adjoint eigenvalue problem (19) and its approximations by Galerkin's method are real numbers, the latter ones are not necessarily upper bounds for $\lambda_n(A)$. This is a remarkable deviation from the circumstances which are encountered when Galerkin's method is applied to self-adjoint eigenvalue problems.

4. RESULTS

In order to find the critical Mach number, M_{crit}, for the flutter problem under consideration, one has to proceed as outlined in the Introduction and then repeatedly discussed in Sections 1 and 3, using the results of Section 3. As has been shown in Section 3.2, we can restrict ourselves in this regard to the consideration of the eigenvalue $\lambda_1(A)$. Most of the calculations have been done for a cylindrical shell of electroplated copper with the ratio $L/R = 2$, since such a cylinder has been used for the experiments. For a_0 and ρ_0 the corresponding values of the standard atmosphere for the elevation of 50,000 ft and for sea level have been employed. Experimental investigations, performed during spring, 1962, at the Graduate Aeronautical Laboratories, indicated that for the damping coefficient γ of the cylinders used, values in the neighborhood of 10^{-3} are reasonable; for these investigations the formulas, presented in [12], have been applied.

Since linear piston theory is certainly not reliable for Mach numbers less than $\sqrt{2}$, we restrict ourselves to the region $M \geq 1{\cdot}5$. Since the eigenvalues of the eigenvalue problem (19) have been investigated only for $0 \leq A \leq 110{,}000$, our following statements are restricted to this interval.

4.1. *The critical Mach number for an unpressurized copper cylinder at 50,000 ft elevation and at sea level*

In Fig. 8 the curves I, II, and III demonstrate the results for an unpressurized copper cylinder with $L/R = 2$ at 50,000 ft elevation. The following constants were used:

$$\left.\begin{aligned} a_0 &= 1{\cdot}164 \times 10^4 \text{ in./sec;} \\ \rho_0 &= 1{\cdot}741 \times 10^{-8} \text{ lb in.}^{-4} \text{ sec}^2; \end{aligned}\right\} \quad \text{(50,000 ft elevation)} \tag{92a}$$

$$\left.\begin{aligned} \rho_s &= 8{\cdot}333 \times 10^{-4} \text{ lb in.}^{-4} \text{ sec}^2; \\ \nu &= 0{\cdot}33; \qquad E = 1{\cdot}3 \times 10^7 \text{ lb in.}^{-2}; \\ p_i &= 0; \qquad \frac{L}{R} = 2. \end{aligned}\right\} \tag{92b}$$

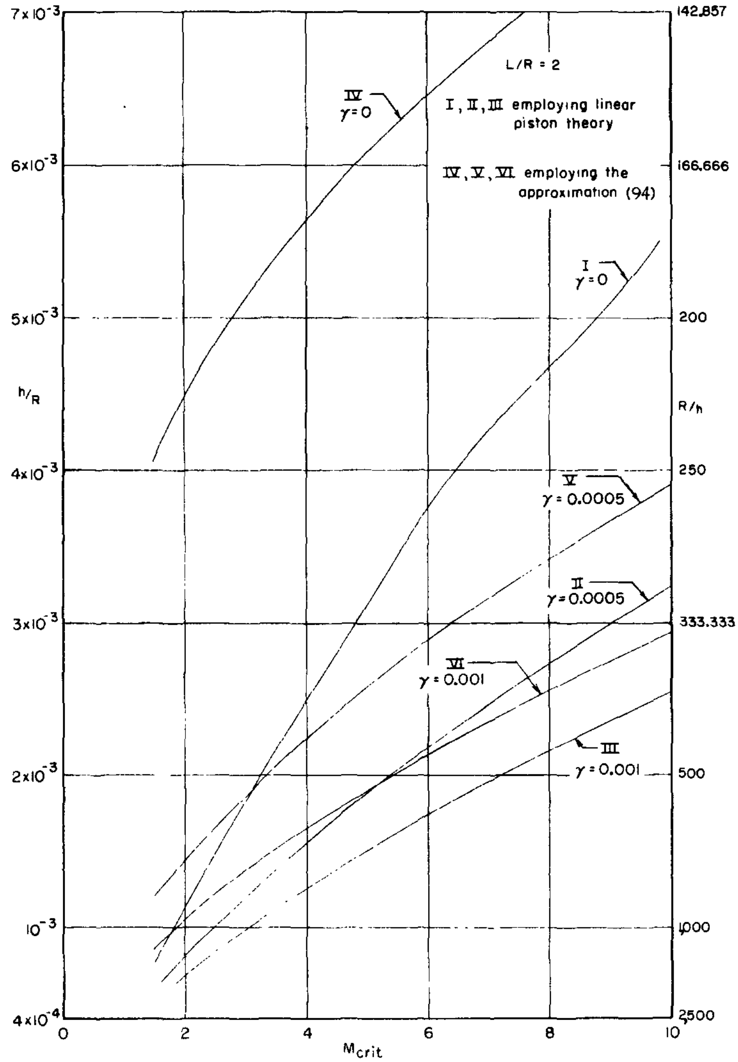

FIG. 8. Flutter boundary (minimum thickness ratio h/R required to prevent flutter versus Mach number) for a simply supported unpressurized copper cylinder for 50,000 ft altitude atmospheric conditions for various damping coefficients γ. Linear piston theory as well as the pressure approximation (94) have been employed.

The curves, called the 'flutter boundaries', show the required thickness ratio h/R in order to prevent flutter in an axisymmetrical mode as a function of the Mach number. The curves, in turn, separate for any thickness ratio the region of the Mach numbers. For all the Mach numbers on the left-side of the flutter boundaries the cylinder is stable; for the Mach number on the boundary (critical Mach number) the cylinder is neutrally stable; for the Mach numbers on the right-side of the curves the shell is unstable, i.e. flutter occurs. (The last statement has to be restricted to values of A not larger than 110,000, see above.)

The curves I($\gamma = 0$), II($\gamma = 0{\cdot}0005$), and III($\gamma = 0{\cdot}001$) illustrate the influence of the damping coefficient γ. For increasing values of γ the flutter boundaries are shifted to the right, i.e. the regions of stable Mach numbers increase. This statement contradicts the corresponding result of Leonhard and Hedgepeth [22]. Curve III could not be continued farther to the left because of lack of investigations of the eigenvalue problem (19) for A-values beyond 110,000. The remaining curves IV, V, and VI in Fig. 8 will be treated in Section 4.2.

Analogous calculations have been carried out for sea level conditions. The reader will find the corresponding results in Fig. 9 of [18].

There is always a question as to the severity of the flutter vibrations, i.e. how fast the amplitude increases, if M is on the right side of the flutter boundary (see e.g. [1] and [23]). In order to check this for the copper cylinder used for the experiments ($R = 8$ in.; $L/R = 2$), the following calculations have been carried out. For some combinations of the thickness ratio h/R and the damping coefficient γ the corresponding value $1{\cdot}01 \times M_{\text{crit}}$ was determined. For this value of M the corresponding value A and finally the corresponding complex eigenvalue $\lambda_1(A)$ were evaluated. From this value $\lambda_1(A)$ the frequency $\omega_{12}(A)$ was determined using equations (20) and (18e) and the value $R = 8$ in. The results, which apply to 50,000 ft elevation are listed below:

$$\begin{aligned} &\frac{h}{R} = 6{\cdot}25 \times 10^{-4}; \qquad \gamma = 0; \qquad &&\omega_{12}(A) = 15{,}627 - i0{\cdot}142;\\ &\frac{h}{R} = 6{\cdot}25 \times 10^{-4}; \qquad \gamma = 0{\cdot}0005; \qquad &&\omega_{12}(A) = 15{,}632 - i0{\cdot}426;\\ &\frac{h}{R} = 6{\cdot}25 \times 10^{-4}; \qquad \gamma = 0{\cdot}001; \qquad &&\omega_{12}(A) = 15{,}636 - i0{\cdot}521. \end{aligned} \tag{93}$$

The thickness ratio of the cylinder used in the experiments is approximately $h/R = 6{\cdot}25 \times 10^{-4}$. The corresponding values for $h/R = 10^{-3}$ are given in [18].

These figures show that, in the frame of the present analysis, the flutter vibration has a rather fast growing amplitude as soon as the Mach number is 1 per cent larger than the critical Mach number under the given conditions. Under the premise that the basic physical assumptions which lead to the eigenvalue problem (19) are satisfactory, one can conclude that flutter must be observed during the experiments as soon as the Mach number crosses the flutter boundary from the left to the right. In other words, if flutter will not occur during the experiments when M is beyond the critical value, then the basic physical assumptions are unsatisfactory.

The influence of the length ratio L/R on the critical Mach number has been investigated too. Here it may suffice to state that the influence of the change of L/R diminishes as L/R increases. The corresponding results are listed in Fig. 10 of [18].

4.2. *The influence of the aerodynamic damping*

Some previous investigations, e.g. Johns [24] and [25] and Shulman [26] claim it to be sufficient to use for the aerodynamic pressure the approximation

$$p_0(x, t) = -a_0 \rho_0 U \frac{\partial \hat{w}}{\partial x} \tag{94}$$

instead of the approximation (7) used in the preceding calculations. The approximation (94) neglects the aerodynamic damping originated by the $\partial/\partial t$-term in (7). Using (94) for the aerodynamic pressure, one has to drop the term $ia_0\rho_0\omega$ in the expressions for λ^* and λ, see (18d) and (18e). Then for $\beta = 0$, i.e. for $\gamma = 0$, the stability parabola (23a) collapses to the non-negative part of the real axis in the λ^*-plane, and hence the vibrations are unstable as soon as $\lambda_1^*(A)$ or $\lambda_1(A)$ becomes complex, i.e. for $A > \dot{A}_2$. For $\gamma > 0$ the analogy to the stability parabola (23b) reads

$$Re(\lambda)+B = \frac{E\rho_s}{12(1-\nu^2)\beta^2}\left(\frac{R}{L}\right)^4\left(\frac{h}{R}\right)^4[\mathrm{Im}(\lambda)]^2. \tag{95}$$

Corresponding calculations were carried out for $\gamma = 0$, $\gamma = 0{\cdot}0005$, and $\gamma = 0{\cdot}001$, using the constants listed under (92). The results are shown by the curves IV, V, and VI in Fig. 8. A comparison with the curves I, II, and III discloses the stabilizing influence of the aerodynamic damping which originates from the $\partial/\partial t$-term in equation (7). In this way it is demonstrated that (in the frame of the present analysis) the aerodynamic damping is not negligible.

It should be noticed in this regard that Voss ([27], pp. 955 and 956) mentioned that the midplane inertia forces, which are neglected in the present analysis (see Section 1), have an additional stabilizing effect.

For the following calculations, again linear piston theory [i.e. the approximation (7)] has always been employed.

4.3. *The influence of the internal pressure p_i*

So far only calculations for the unpressurized cylinder, i.e. $p_i = 0$, have been carried out. In order to check the influence of the internal pressure differential p_i, calculations have been performed, again employing linear piston theory and using the constants (92), but

$$p_i = 5\ \mathrm{lb\ in}^{-2}. \tag{96}$$

The results did not disclose any significant influence of pressurizing on the flutter boundary.

4.4. *Comparison with previous results*

Recently, Holt and Strack [4] investigated the flutter of a cylindrical shell of finite length, and *inter alia* the flutter in an axisymmetrical mode. (In their numerical work, Holt and Strack introduced a 'quasi-steady' assumption in the aerodynamics which is further simplified by an approximate kernel.) A comparison of their results with those obtained by the present analysis (again employing linear piston theory) demonstrates that the present analysis leads to considerably higher critical Mach numbers than Holt and Strack's analysis. More details are presented in Section 4.4 of [18].

4.5. *Comparison with experiments*

An electroplated copper cylinder was used for a flutter experiment performed in the 8 × 7 ft supersonic section of the Unitary Plan Wind Tunnel of the NASA Ames Laboratory in June 1961 (see [2] and [3]). The measurements of the cylinder were

$$L = 16\ \mathrm{in.};\qquad R = 8\ \mathrm{in.};\qquad h/R = 6{\cdot}25\times 10^{-4}. \tag{97}$$

The tunnel was adjusted to the 50,000 ft altitude atmospheric conditions, see (92a). No flutter could be observed in the range $2{\cdot}5 \leq M \leq 3{\cdot}5$, contrary to the results obtained by the present analysis mentioned in Section 4.1, see Fig. 8.

This statement concerning the disagreement of the theoretical and the experimental results is based upon the assumption that the damping coefficient γ is in the neighborhood of $\gamma = 10^{-3}$. This assumption is backed by experiments which were performed at the Graduate Aeronautical Laboratories in the early months of 1962. In these investigations the damping coefficient γ was determined by a decaying-free vibration of the cylinders used for the Ames-tests, employing the formulas of [12]. During the damping experiments mentioned above, vibration modes with several nodal lines parallel to the cylinder-axis have been excited, the corresponding frequencies are of the order of 10^2. Unfortunately, no axisymmetrical decaying-free vibrations could be investigated during these tests; it is noteworthy that the axisymmetrical flutter mode obtained by the present analysis have frequencies of the order of 10^3, see e.g. (93). Hence it is not perfectly clear whether or not the result $\gamma \approx 10^{-3}$, which was obtained from the damping tests, can properly be applied to the axisymmetrical flutter problem under investigation. However, no more illuminating results concerning γ are available.—In case the proper value for γ to be employed for the present problem should be in the neighborhood of $\gamma = 5{\cdot}10^{-3}$ or even larger, then preliminary calculations demonstrate that the present theory would *not* predict flutter (in an axisymmetrical mode) of the tested cylinders in the range $2{\cdot}5 \leq M \leq 3{\cdot}5$.

5. COMPARISON OF MOVCHAN'S ANALYSIS WITH THE PRESENT ANALYSIS, STEPANOV'S RESULTS

At this final stage some remarks concerning the relation between Movchan's analysis and the present one and a short discussion of Stepanov's results are worthwhile.

As mentioned already in the Introduction and in several footnotes, Movchan considered in [5–7] the eigenvalue problem (19) and similar ones. (Refs. [6] and [7] came to the knowledge of the author after the present analysis had already been carried out.) The flutter of a flat cantilever panel of infinite width and of a rectangular flat panel under edge compression, which lead to eigenvalue problems similar to (19), has been investigated in [5] and [6] respectively. The basic analysis is developed in [5]. In contrast to the present analysis, Movchan starts with the investigation of the solutions of a characteristic system which is analogous to the system (67), employing the theorem of implicit functions and Weierstrass' lemma (Weierstrasso'schen Vorbereitungssatz, see [20], p. 126). In this way he proves the continuous dependence of the branches $(\alpha_n(A), \beta_n(A))$ upon the generalized velocity A and their behavior on branch points. Some details are left to the reader. Remarks analogous to the investigations in the preceding Sections 2.2 and 2.4 are not given. The fact that the eigenvalues $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ become complex as soon as the curve $\mathfrak{C}_A^{\mp}$ no longer has real intersections with the loop l_{2m} is demonstrated in [6], pp. 8 and 9, using Weierstrass' lemma. For non-real intersections of $\mathfrak{C}_A^{\pm}$ with the loop l_{2m}, approximate expressions for $\lambda_{2m-1}(A)$ and $\lambda_{2m}(A)$ are given in [6], which are satisfactory for relatively small values of A. In [7] finally complex eigenvalues are evaluated for values of A up to $A = 9000$.

The investigation of the interval $0 \leq A \leq 9000$ is sufficient for the study of the flutter of a flat panel. Here the constant B equals zero, see Section 1. But for a cylindrical shell, B is a rather large positive number, and therefore the eigenvalues $\lambda_n(0)$ are far more remote from the vertex of the stability parabola (23b). Accordingly, in the case of a cylindrical shell much larger values of A are required to make $\lambda_1(A)$ intersect the stability parabola. (In this respect see also the remarks in Section 4.1 concerning curve III in Fig. 8).

In [28], Stepanov studies *inter alia* the flutter of a simply supported cylinder of finite length, applying Movchan's results given in [5] and [6]. Stepanov presents critical Mach numbers for flutter modes with four circumferential waves, which are reproduced in Fig. 18 of [1]. His critical Mach numbers are evaluated from the value $A = \mathring{A}_2 = 343{\cdot}3$, for which the eigenvalues $\lambda_1(A)$ and $\lambda_2(A)$ coincide, see especially [28], pp. 17 and 20. (According to some numerical checks of the figures listed in Table 1 of [28], it seems to be most likely that even a smaller value of A, close to $A = A_2^* = 80\pi^3/(3\sqrt{6})$, see the list (82), has been used in Stepanov's analysis.) That means, according to the remarks in Section 4.2, that Stepanov's critical Mach numbers are evaluated under the assumption that the aerodynamic damping is negligible.

6. CONCLUSIONS

The flutter problem of a simply supported cylindrical shell of finite length in an axisymmetric mode in a supersonic airstream has been studied. Using the linearized Timoshenko shell equations, linear piston theory, and neglecting the inertia forces of the shell elements in the longitudinal direction, the flutter problem was reduced to a non-self-adjoint eigenvalue problem of an ordinary differential equation of the fourth order. This eigenvalue problem was solved in a rigorous manner. The results show that, in the framework of this analysis, the aerodynamic damping as well as the structural damping has a considerable stabilizing effect. The influence of the internal static pressure differential is very small. As soon as the velocity of the airstream is a little bit beyond the evaluated flutter speed the amplitude of the flutter mode grows rather fast.

A comparison of the 'exact' results obtained by the present analysis and those yielded by Galerkin's method shows that Galerkin's method leads to dependable results in applying it to the non-self-adjoint eigenvalue problem under consideration. However, the larger the value of the generalized velocity A, the more modes one has to employ to obtain reliable answers.

Experimental results were used as a touchstone for the preceding investigations. During the tests no flutter could be observed in the Mach number range predicted by the present analysis. This indicates that the physical assumptions on which the establishment of the eigenvalue problem was based are not satisfactory. The approximation of the aerodynamic pressure by the linear piston theory expression, the disregarding of boundary layer effects, and the neglecting of the inertia forces of the shell elements in the longitudinal direction might be responsible at least partially for this disagreement.

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Zusammenfassung—In dieser Arbeit wird das Flattern mit axialsymmetrischer Schwingungsform eines an beiden Enden frei aufliegenden dünnwandigen Zylinders untersucht. Die linearisierten Schalengleichungen von Timoshenko und die Anwendung der linearen Piston-Theorie führen zu einem nicht-selbstadjungierten Eigenwertproblem, das eingehend untersucht und ohne weitere Approximationen streng gelöst wird. Auf diese Weise werden die kritischen Geschwindigkeiten (Flatter-Grenzen) bestimmt. Unter Benutzung der so gewonnenen 'exakten' Ergebnisse können zwei Fragen nachgeprüft werden. Ein Vergleich dieser exakten Resultate mit den mittels der Galerkin'schen Methode gefundenen Näherungen zeigt, dass diese Methode bei Anwendung auf das vorliegende Eigenwertproblem zu zuverlässigen Ergebnissen führt. Anderseits kann man an Hand eines Vergleiches der exakten Resultate mit Versuchsergebnissen entscheiden, ob die dem Eigenwertproblem zugrunde liegenden physikalischen Annahmen ausreichend sind.

Абстракт—В этой работе изучается флаттер в осе-симметричной форме колебания свободно опирающейся тонкой цилиндрической оболочке. Линейно-преображенные уравнения Тимошенко для оболочек и теория линейных поршней приводят к задаче о несамосопряженных собственных значениях, которая тщательно исследуется и решается без дальнейших приближений. Таким образом определяются критические скорости (границы флаттера). Можно сделать две проверки, употребляя эти точные результаты. Сопоставление точных результатов с результатами полученными по методу Галеркина доказывает надежность этого метода в приложении к настоящей задаче о собственных значениях. С другой стороны, сравнение точных результатов с экспериментальными данными покажет удовлетворительны-ли физические предположения на которых основана задача о собственных значениях.